

HOMEWORK 7, CALCULUS III

①

§16.2 #6

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \int_{-1}^5 \cos(y) dx dy &= \int_{\pi/6}^{\pi/2} \left(\cos(y) \times \left| \begin{array}{l} x = 5 \\ x = -1 \end{array} \right. \right) dy \\ &= \int_{\pi/6}^{\pi/2} 6 \cos(y) dy \\ &= 6 \sin(y) \Big|_{\pi/6}^{\pi/2} \\ &= 6 \sin(\pi/2) - 6 \sin(\pi/6) \\ &= 6 - 3 \\ &= \boxed{3} \end{aligned}$$

§16.2 #10

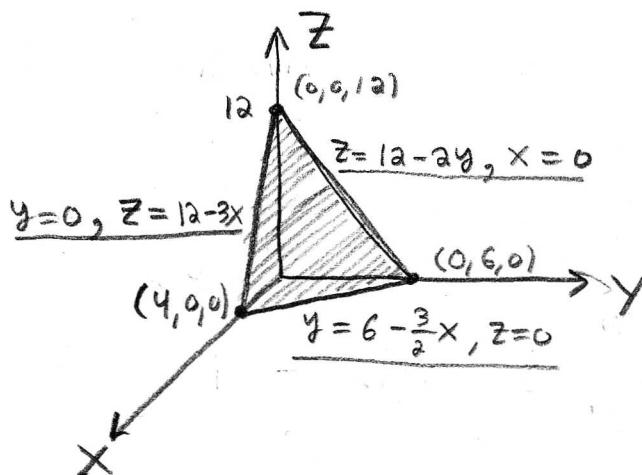
$$\begin{aligned} \int_0^1 \int_0^3 e^{x+3y} dx dy &= \int_0^1 \int_0^3 e^x e^{3y} dx dy \\ &= \int_0^1 e^{3y} \int_0^3 e^x dx dy \\ &= \int_0^1 e^{3y} \left(e^x \Big|_{0=x}^3 \right) dy \\ &= \int_0^1 (e^3 - 1) e^{3y} dy \\ &= \left[\frac{e^3 - 1}{3} \right] e^{3y} \Big|_0^1 \\ &= \left(\frac{e^3 - 1}{3} \right) (e^3 - 1) \\ &= \boxed{\frac{1}{3} (e^3 - 1)^2} \end{aligned}$$

{ Only can
do this
when bounds
are constants }

②
 §16.2 # 25] Find volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle

$$R = \{(x, y) \mid 0 \leq x \leq 1, -2 \leq y \leq 3\}.$$

We ought to integrate $Z = 12 - 3x - 2y \equiv f(x, y)$ on R . This gives the sum of volumes with height Z . Well, let's be careful, it gives the signed volume hopefully $f(x, y) \geq 0$ for $(x, y) \in R$.



Graph $Z = 12 - 3x - 2y$ to be safe.

$$x = 0, z = 12 - 2y$$

$$y = 0, z = 12 - 3x$$

$$z = 0, 0 = 12 - 3x - 2y$$

$$y = 6 - \frac{3}{2}x$$

As you can see the graph $Z = 12 - 3x - 2y$ is entirely above the xy -plane for the region described, $0 \leq x \leq 1$ with $-2 \leq y \leq 3$ puts $Z = f(x, y) > 0$.

You can argue that the analysis above is not needed because the problem states the volume lies above R and below $Z = 12 - 3x - 2y$. That's ok, you didn't have to do this graph, BUT it would be reasonable to ask you to do such an argument. Can you follow it?

§ 16.2 #25 | The preceding page showed this is a reasonable calculation
to find the volume of the solid. 3

$$\begin{aligned} V &= \iint_R f \, dA \\ &= \int_{-2}^3 \int_0^1 (12 - 3x - 2y) \, dx \, dy \\ &= \int_{-2}^3 \left[12x - \frac{3}{2}x^2 - 2yx \right] \Big|_0^1 \, dy \\ &= \int_{-2}^3 \left[12 - \frac{3}{2} - 2y \right] \, dy \\ &= \int_{-2}^3 \left(\frac{21}{2} - 2y \right) \, dy \\ &= \left(\frac{1}{2}21y - y^2 \right) \Big|_{-2}^3 \\ &= \left(\frac{1}{2}(21)(3) - 9 \right) - \left(-\frac{42}{2} - 4 \right) \\ &= \left(63/2 - 18/2 \right) - (-21 - 4) \\ &= 45/2 + 25 \\ &= 45/2 + 50/2 \\ &= \boxed{95/2} \end{aligned}$$

§16.6 #3

(4)

$$\begin{aligned}
 \int_0^1 \int_0^3 \int_0^{x+z} 6xyz \, dy \, dx \, dz &= \int_0^1 \int_0^3 \left(6xyz \Big|_{y=0}^{y=x+z} \right) dx \, dz \\
 &= \int_0^1 \int_0^3 6x(x+z)z \, dx \, dz \\
 &= \int_0^1 \int_0^3 (6x^2z + 6x^2z^2) \, dx \, dz \\
 &= \int_0^1 \left(2x^3z + 3x^2z^2 \Big|_{x=0}^{x=z} \right) dz \\
 &= \int_0^1 (2z^4 + 3z^2) \, dz \\
 &= \int_0^1 5z^4 \, dz \\
 &= z^5 \Big|_0^1 \\
 &= \boxed{1}
 \end{aligned}$$

§16.6 #10 Let $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, x \leq z \leq 2x\}$

$$\begin{aligned}
 \iiint_E xyz \cos(x^5) \, dV &= \int_0^1 \left(\int_x^{2x} \left(\int_0^x yz \cos(x^5) \, dy \right) dz \right) dx \\
 &= \int_0^1 \int_x^{2x} \left(z \cos(x^5) \frac{y^2}{2} \Big|_0^x \right) dz \, dx \\
 &= \int_0^1 \int_x^{2x} \left(\frac{1}{2} x^2 z \cos(x^5) \right) dz \, dx \\
 &= \int_0^1 \left(\frac{1}{2} x^2 \cos(x^5) \frac{z^2}{2} \Big|_x^{2x} \right) dx \\
 &= \int_0^1 \left(\frac{1}{2} x^2 \cos(x^5) \frac{1}{2} (4x^2 - x^2) \right) dx \\
 &= \int_0^1 \frac{3}{4} x^4 \cos(x^5) \, dx
 \end{aligned}$$

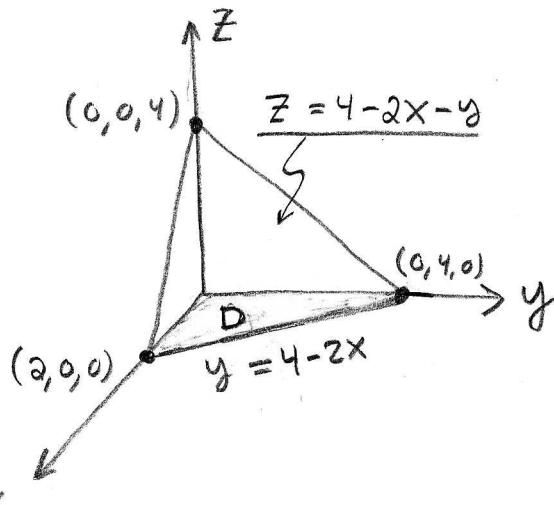
$$u = x^5 \quad du = 5x^4 dx$$

$$u(1) = 1, u(0) = 0$$

$$\begin{aligned}
 &= \int_0^1 \frac{3}{20} \cos(u) \, du \\
 &= \frac{3}{20} (\sin(1) - \sin(0)) = \boxed{\frac{3 \sin(1)}{20}}
 \end{aligned}$$

(5)

§16.6 #19 Tetrahedron enclosed by coordinate planes
 $2x + y + 3z = 4$. Look at intersection with coord. planes



Observe from graph,

$$D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 4 - 2x\}$$

Thus

$$\begin{aligned}
 V &= \iint_D (4 - 2x - y) dA \\
 &= \int_0^2 \int_0^{4-2x} (4 - 2x - y) dy dx \\
 &= \int_0^2 \left((4 - 2x)y - \frac{1}{2}y^2 \Big|_0^{4-2x} \right) dx \\
 &= \int_0^2 \left((4 - 2x)(4 - 2x) - \frac{1}{2}(4 - 2x)^2 \right) dx \\
 &= \int_0^2 \frac{1}{2}(4 - 2x)^2 dx \\
 &= \int_0^2 \frac{1}{2}(16 - 16x + 4x^2) dx \\
 &= \int_0^2 (8 - 8x + 2x^2) dx \\
 &= \left(8x - 4x^2 + \frac{2}{3}x^3 \Big|_0^2 \right) = 16 - 16 + \left(\frac{2}{3}\right)8 = \boxed{\frac{16}{3}}
 \end{aligned}$$

Gather information,

$$x = 0, z = 4 - y$$

$$y = 0, z = 4 - 2x$$

$$z = 0, y = 4 - 2x$$

$$x = y = 0, (0, 0, 4)$$

$$x = z = 0, (0, 4, 0)$$

$$y = z = 0, (2, 0, 0)$$

The facts above help make the graph.