

BASIC LIMITS HOMEWORK SOL

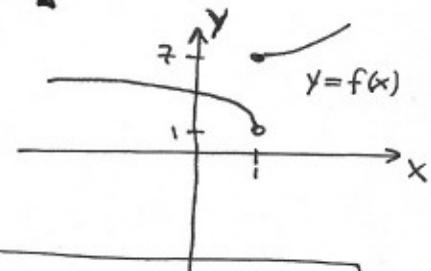
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§2.2 #2 Explain what it means to say $\lim_{x \rightarrow 1^-} f(x) = 3$ and $\lim_{x \rightarrow 1^+} f(x) = 7$.
In this case does $\lim_{x \rightarrow 1} f(x)$ exist?

$\lim_{x \rightarrow 1^-} f(x) = 3$ means that as we take values of $x < 1$ which get closer and closer to 1 we find $f(x)$ gets closer and closer to 3. ("left limit")

$\lim_{x \rightarrow 1^+} f(x) = 7$ means that as we take values of $x > 1$ which get closer and closer to 1 we find $f(x)$ gets closer and closer to 7. ("right limit")

By definition $\lim_{x \rightarrow 1} f(x)$ exists only if both the left and right limits exist and are equal. Here is a possible example ↗



§2.3 #4 P. 117 Calculate the limit below using the tools from the LECTURE notes. (no limit laws necessary, rather use the Thm)

$$\lim_{x \rightarrow 2} \left(\frac{2x^2 + 1}{x^2 + 6x - 4} \right) = \frac{2(2)^2 + 1}{(2)^2 + 6(2) - 4} = \frac{9}{12} = \boxed{\frac{3}{4}}$$

Noting that $2 \in \text{dom} \left(\frac{2x^2 + 1}{x^2 + 6x - 4} \right)$ and $f(x) = \frac{2x^2 + 1}{x^2 + 6x - 4}$ is a rational function which is continuous on its domain.

§2.3 #6 P. 117 Calculate limit using LECTURE NOTES.

$$\lim_{x \rightarrow -2} \sqrt{u^4 + 3u + 6} = \sqrt{(-2)^4 - 6 + 6} = \sqrt{16} = \boxed{4}$$

Clearly -2 is in the domain of $f(u) = \sqrt{u^4 + 3u + 6}$. (Whenever we can simply evaluate the function at the limit point it is because the limit point is in the domain of the continuous function.)

§2.3 #8 P. 117 This problem is subtle, but it strikes at the very heart of why limits are useful.

a.) $\frac{x^2 + x - 6}{x - 2} = \frac{(x+3)(x-2)}{(x-2)} = x+3$ • this is only true when $x \neq 2$. We cannot cancel $(x-2)$ with $(x-2)$ when $x=2$ because of division by zero.

b.) $\lim_{x \rightarrow 2} \left(\frac{x^2 + x - 6}{x - 2} \right) = \lim_{x \rightarrow 2} \left(\frac{(x+3)(x-2)}{(x-2)} \right) = \lim_{x \rightarrow 2} (x+3) = 5.$

• it's ok to cancel the $(x-2)$'s inside the limit because we let x get close to 2 in the limit, BUT $x \neq 2$ inside the limit!

§2.3 #14
P. 118

Use algebra to determine the limit

• you cannot just stick in $h = 0$ because $\frac{\sqrt{1+h} - 1}{h} = \frac{0}{0}$ which is undefined.
the function is not defined at $h = 0$!

$$\begin{aligned}\lim_{h \rightarrow 0} \left(\frac{\sqrt{1+h} - 1}{h} \right) &= \lim_{h \rightarrow 0} \left(\left(\frac{\sqrt{1+h} - 1}{h} \right) \left(\frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \right) \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{(\sqrt{1+h})^2 + \sqrt{1+h} - \sqrt{1+h} - 1}{h(\sqrt{1+h} + 1)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{h}{h(\sqrt{1+h} + 1)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{\sqrt{1+h} + 1} \right) \\ &= \frac{1}{\sqrt{1+0} + 1} = \boxed{\frac{1}{2}}\end{aligned}$$

why can I cancel
the h inside
the limit? Could
I do it outside
the limit in general?

• neat, the function $f(h) = \frac{\sqrt{1+h} - 1}{h}$
is not defined at $h = 0$, but
it has a limit at that value.

§2.3 #20
P. 118

Use algebra to resolve the indeterminacy of
the limit below

(LATER WE'LL TALK MORE
ABOUT HOW THIS LIMIT
HAS TYPE $0/0$)

$$\begin{aligned}\lim_{t \rightarrow 0} \left[\frac{1}{t} - \frac{1}{t^2+t} \right] &= \lim_{t \rightarrow 0} \left[\frac{t^2+t - t}{t(t^2+t)} \right] : \text{made a common denominator.} \\ &= \lim_{t \rightarrow 0} \left[\frac{t^2}{t^2(t+1)} \right] : \text{why can I cancel those } t^2 \text{ terms here?} \\ &= \lim_{t \rightarrow 0} \left[\frac{1}{t+1} \right] \\ &= \boxed{1}\end{aligned}$$

§2.3 #26 (Use the squeeze Thⁿ)
P. 118 Given that $3x \leq f(x) \leq x^3 + 2$ for all x
find then the $\lim_{x \rightarrow 1} f(x)$.

We know that $f(x)$ is sandwiched in between $3x$ and $x^3 + 2$. Notice,

$$\lim_{x \rightarrow 1} (3x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1} (x^3 + 2) = 3$$

Thus by the Squeeze Thⁿ $f(x)$ must have the same limit,

$$\boxed{\lim_{x \rightarrow 1} f(x) = 3}$$

(Don't even need to know an explicit formula for $f(x)$ the fact that it's squeezed between $3x$ and $x^3 + 2$ determines the limit.)

§2.3 #38

p. 118

In relativity a rigid rod of length L_0 (in its rest frame) is observed to have length L in some other frame according to the "Lorentz contraction" formula, (γ is same as in pg. 5 of h/w)

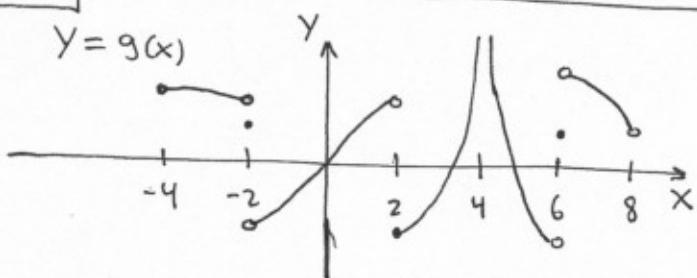
$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The question is what length L does the rod have when it is moving very fast, say as $v \rightarrow c$ from the left, and why do we need a left limit?

$$\lim_{v \rightarrow c^-} (L) = \lim_{v \rightarrow c^-} \left(L_0 \sqrt{1 - \frac{v^2}{c^2}} \right) = L_0 \lim_{v \rightarrow c^-} \sqrt{1 - \frac{v^2}{c^2}} = L_0 \underbrace{\sqrt{1 - \frac{c^2}{c^2}}}_{\text{Zero!}} = 0$$

the length becomes zero as we approach the speed of light. We need a left limit because we cannot go faster than the speed of light. That is reflected in the fact that $\text{dom}(\sqrt{1 - \frac{v^2}{c^2}}) = [-c, c]$, if you put in $v > c$ you will get an imaginary #. The speed limit $v \leq c$ is built into the equations of relativity.

Bonus Point: why is $L = \frac{L_0}{\gamma}$ not quite true mathematically speaking?

§2.4 #4
p. 128From the graph state the intervals on which g is continuous

The function is continuous wherever its value is equal to its limit. At endpoints we adopt the convention of using one-sided limits, like at $x = -4$ the function is continuous because $\lim_{x \rightarrow -4^+} (g(x)) = g(-4)$.

ANSWER: $[-4, -2) \cup (-2, 2) \cup (2, 4) \cup (4, 6) \cup (6, 8)$ the "U" means union.

§2.4 #18
p. 129

Find the domain and explain why the function is continuous. (Use LECTURE NOTES)

$f(t) = 2t + \sqrt{25 - t^2}$ ← this is an algebraic function we know these are continuous on their domains.

When does the formula above not make sense? We need that $\sqrt{\text{positive (or zero)}}$

$$25 - t^2 \geq 0$$

$$25 \geq t^2$$

$$-5 \leq t \leq 5 \quad \leftarrow \text{this gives the domain}$$

$$\text{dom}(f) = [-5, 5]$$

§ 2.4 #26
p. 129

Use that $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$ when $(\lim_{x \rightarrow a} g(x)) \in \text{dom}(f)$
and f and g are continuous

$$\begin{aligned}\lim_{x \rightarrow \pi} (\sin(x + \sin(x))) &= \sin \left[\lim_{x \rightarrow \pi} (x + \sin(x)) \right] \\ &= \sin \left[\lim_{x \rightarrow \pi} x + \lim_{x \rightarrow \pi} \sin(x) \right] \\ &= \sin \left[\pi + \sin_0(\pi) \right] \\ &= \sin(\pi) = \boxed{0}\end{aligned}$$

§ 2.4 #30
p. 129

The gravitational force exerted by Earth on a unit mass at a distance r from center of Earth is given by:

$$F(r) = \begin{cases} \frac{GMr}{R^3} & \text{if } r < R \\ \frac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M = mass of earth, R = radius of earth and G is the gravitational constant. Is the force a continuous function of r ?

If $r < R$ then $F(r) = \frac{GMr}{R^3}$ which is a linear function of r hence its continuous. If $r > R$ then $F(r) = \frac{GM}{r^2}$ which is actually a power function ($F(r) = GMr^{-2}$) so its continuous. The only non-obvious thing is whether the functions agree at $r=R$. That is does $\lim_{r \rightarrow R} F(r) = F(R) = \frac{GM}{R^2}$? How do we verify that?

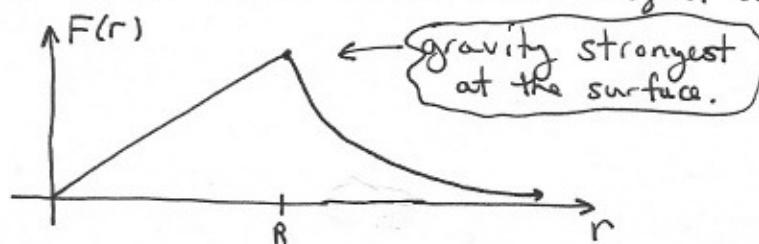
Simple, check the right & left limits at R ,

$$\lim_{r \rightarrow R^-} F(r) = \lim_{r \rightarrow R^-} \left[\frac{GMr}{R^3} \right] = \frac{GMR}{R^3} = \frac{GM}{R^2}$$

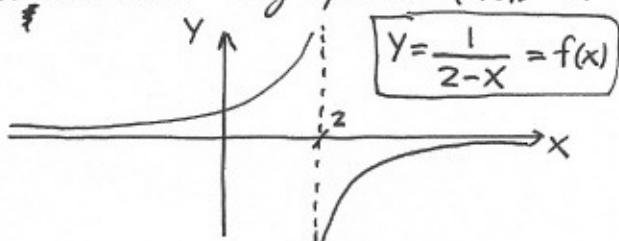
$$\lim_{r \rightarrow R^+} F(r) = \lim_{r \rightarrow R^+} \left[\frac{GM}{r^2} \right] = \frac{GM}{R^2} \therefore \lim_{r \rightarrow R} F(r) = \frac{GM}{R^2} \quad \checkmark$$

Thus $\lim_{r \rightarrow a} F(r) = F(a)$ for all $a \in \mathbb{R}$ so F is continuous.

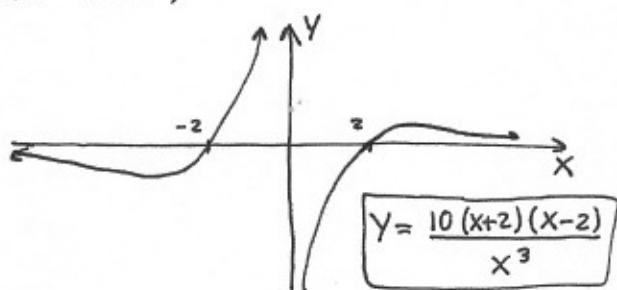
Remark: this is a crude model of the force, it assumes the earth has uniform density, like a big piece of cheese. The real story is more complicated and still a controversial subject for many scientists.



- a.) No $y = f(x)$ cannot intersect a vertical asymptote. If it did then it would have some finite value, but then it wouldn't be an asymptote would it? YES $y = f(x)$ can intersect a horizontal asymptote (lots of times even)

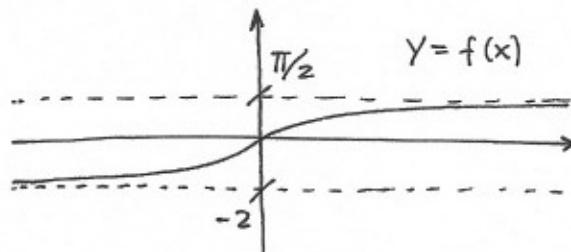


function undefined at V.A.
 $x \notin \text{dom}(f)$



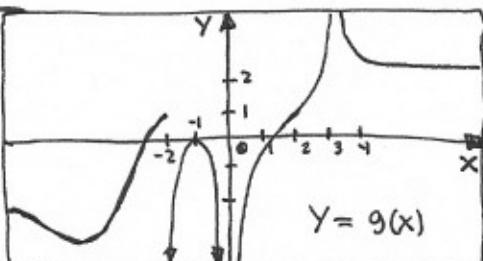
H.A. at $\pm\infty$ is $y = 0$
notice $y = 0$ for $y = f(x)$
at $x = 2$ and $x = -2$.

- b.) A function can have at most 2 horizontal asymptotes, one at ∞ and another at $-\infty$. (In the graph above they were the same.)



$$f(x) = \begin{cases} \tan^{-1}(x) & x \geq 0 \\ \frac{-2x}{x+2} & x < 0 \end{cases}$$

Of course most functions do not even have horizontal asymptotes eg, $x, x^2, \sin(x), \cos(x), e^x, \tan(x), \cosh(x), \sinh(x), \dots$



a.) $\lim_{x \rightarrow \infty} [g(x)] = 2$

d.) $\lim_{x \rightarrow 0} [g(x)] = -\infty$

b.) $\lim_{x \rightarrow -\infty} [g(x)] = -2$

e.) $\lim_{x \rightarrow -2^+} [g(x)] = -\infty$

c.) $\lim_{x \rightarrow 3} [g(x)] = \infty$

f.) parts a.) and b.) reveal the asymptotes are

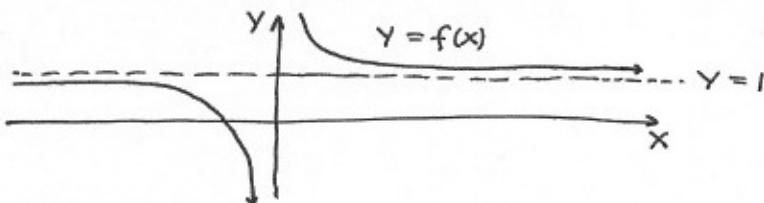
$$y = 2 \quad (\text{as } x \rightarrow \infty)$$

$$y = -2 \quad (\text{as } x \rightarrow -\infty)$$

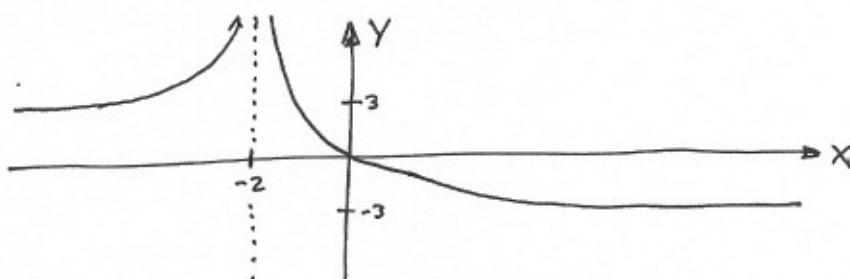
§2.5 #6 Sketch a graph of a function f which satisfies the following criteria:
 p. 140 $\lim_{x \rightarrow 0^+} f(x) = \infty$ & $\lim_{x \rightarrow 0^-} f(x) = -\infty$ & $\lim_{x \rightarrow \infty} f(x) = 1$ & $\lim_{x \rightarrow -\infty} f(x) = 1$

A simple function which does all this is

$$f(x) = \frac{1}{x} + 1$$



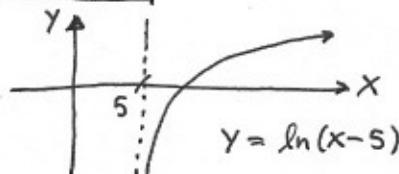
§2.5 #8 Find f with $\lim_{x \rightarrow -2} f(x) = \infty$, $\lim_{x \rightarrow -\infty} f(x) = 3$ and $\lim_{x \rightarrow \infty} f(x) = -3$ and graph it.
 p. 140



I'll let you think about what explicit formula would have this graph.

§2.5 #16 Use your knowledge of basic functions to find limit
 p. 140

$$\lim_{x \rightarrow 5^+} [\ln(x-5)] = -\infty \quad \text{because}$$



(Graph of $y = \ln(x)$ shifted 5 to the right.)

- More generally if we have a limit which puts small positive values into the natural log then we will get zero. This is an important idea cause it's not always easy to graph $f(x)$, for example

$$\lim_{x \rightarrow 0^+} [\ln(\sin(x))] = -\infty \quad \text{because } \sin(x) \rightarrow 0 \text{ as } x \rightarrow 0^+$$

§2.5 #18 Use algebra to remove indeterminacy that is find the limit
 p. 140

$$\lim_{x \rightarrow \infty} \left[\frac{3x+5}{x-4} \right] = \lim_{x \rightarrow \infty} \left[\frac{3 + \frac{5}{x}}{1 - \frac{4}{x}} \right] : \text{dividing top \& bottom by } x.$$

$$= \lim_{x \rightarrow \infty} [3] : \text{because } \frac{1}{x} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

$$= \boxed{3}$$

§2.5 #20
p. 140 Use algebra to find limit

$$\lim_{t \rightarrow -\infty} \left(\frac{t^2 + 2}{t^3 + t^2 - 1} \right) = \lim_{t \rightarrow -\infty} \left(\frac{\frac{1}{t} + \frac{2}{t^3}}{1 + \cancel{\frac{1}{t}} - \cancel{\frac{1}{t^3}}} \right) \quad : \text{dividing top \& bottom by } t^3.$$

$$= \lim_{t \rightarrow -\infty} \left(\frac{\frac{1}{t} + \frac{2}{t^3}}{1 + \cancel{\frac{1}{t}} - \cancel{\frac{1}{t^3}}} \right) \quad : \text{noting the } \frac{1}{t} \text{ and } \frac{1}{t^3} \text{ vanish as } t \rightarrow -\infty.$$

$$= \boxed{0}$$

§2.5 #22
p. 140 Use algebra to find limit

$$\lim_{x \rightarrow \infty} \left(\frac{x+2}{\sqrt{9x^2+1}} \right) = \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{2}{x^2}}{\frac{1}{x} \sqrt{9x^2+1}} \right) \quad : \text{dividing top \& bottom by } x.$$

$$= \lim_{x \rightarrow \infty} \left(\frac{\frac{1}{x^2}}{\sqrt{9 + \frac{1}{x^2}}} \right) \quad : \text{bringing } \frac{1}{x} \text{ inside sqrt makes it into } \frac{1}{x^2} \text{ because } \frac{1}{x} = \sqrt{\frac{1}{x^2}}.$$

$$= \boxed{\frac{1}{3}} \quad : \text{noting } \frac{1}{x^2} \rightarrow 0 \text{ as } x \rightarrow \infty.$$