

Calculus III, Homework 12

①

§ 17.5 #3]  $\vec{F} = \langle 1, x+yz, xy - \sqrt{z} \rangle$  find  $\nabla \times \vec{F} \neq \nabla \cdot \vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy - \sqrt{z} \end{vmatrix}$$

$$= \hat{i} [\partial_y(xy - \sqrt{z}) - \partial_z(x+yz)]$$

$$\hookrightarrow -\hat{j} [\partial_x(xy - \sqrt{z}) - \partial_z(1)] + \hat{k} [\partial_x(x+yz) - \partial_y(1)]$$

$$= \hat{i}(x-y) - \hat{j}(y) + \hat{k}(1)$$

$$= \boxed{\langle x-y, -y, 1 \rangle} = \text{curl}(\vec{F}) = \nabla \times \vec{F}$$

$$\begin{aligned} \nabla \cdot \vec{F} &= (\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}) \cdot (\hat{i} + (x+yz)\hat{j} + (xy - \sqrt{z})\hat{k}) \\ &= \partial_x(1) + \partial_y(x+yz) + \partial_z(xy - \sqrt{z}) \\ &= 0 + 0 + z + 0 - \frac{1}{2\sqrt{z}} \\ &= \boxed{z - \frac{1}{2\sqrt{z}}} = \text{div}(\vec{F}) = \nabla \cdot \vec{F} \end{aligned}$$

Remark: I've used the compact notation

$$\frac{\partial}{\partial x} = \partial_x, \frac{\partial}{\partial y} = \partial_y, \frac{\partial}{\partial z} = \partial_z$$

also you can see we can think of  $\nabla$  as a vector of operators  $\nabla = \hat{i} \partial_x + \hat{j} \partial_y + \hat{k} \partial_z$ .

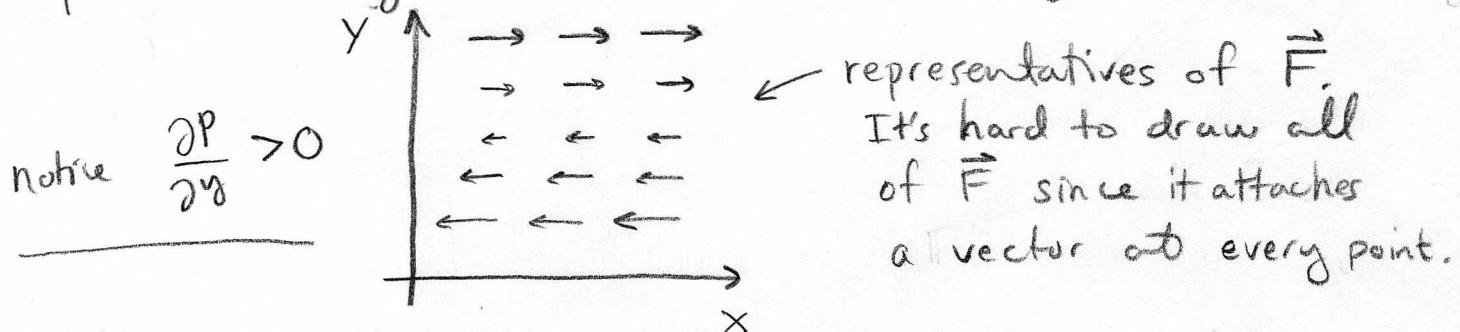
But, be careful, the ordering matters, if  $\vec{F} = \langle F_1, F_2, F_3 \rangle$

$\nabla \cdot \vec{F}$  is a function  $\partial_x F_1 + \partial_y F_2 + \partial_z F_3$ .

$\vec{F} \cdot \nabla = F_1 \partial_x + F_2 \partial_y + F_3 \partial_z$  is an operator.

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§ 17.5 #11 Assume  $\vec{F}$  is a vector field with zero  $z$ -component. This means  $\vec{F} = \langle P, Q, 0 \rangle$ . Moreover, assume  $\vec{F}$  looks the same in all horizontal ( $z = z_0$ ) planes. Analyze  $\text{curl}(\vec{F})$  and  $\text{div}(\vec{F})$  given the picture as well,



It is clear that  $\vec{F} = \langle P, 0, 0 \rangle$ , it has no  $y$ -component.

Moreover, it is clear that  $P$  is a function of  $y$  alone.

thus  $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial z} = 0$ . Thus

$$\begin{aligned} \boxed{\text{curl}(\vec{F})} \quad \nabla \times \vec{F} &= (\hat{i} \partial_x + \hat{j} \partial_y + \hat{k} \partial_z) \times (P \hat{i}) \\ &= (\underbrace{\hat{i} \times \hat{i}}_{\text{zero}}) \partial_x(P) + (\hat{j} \times \hat{i}) \partial_y(P) + (\underbrace{\hat{k} \times \hat{i}}_{\text{zero}}) \partial_z(P) \\ &= \hat{k} (\partial_y P) = \underline{\text{curl}(\vec{F})} \end{aligned}$$

So  $\text{curl}(\vec{F}) \neq 0$ , it points down in the  $z$ -direction

$$\boxed{\text{div}(\vec{F})} \quad \nabla \cdot \vec{F} = \partial_x(P) \quad \text{since } \vec{F} = \langle P, 0, 0 \rangle$$

$= 0$  since the  $\vec{F}$  is the same for each  $x$  once we fix a particular  $y$ -value.

Remark: again I'm trying to illustrate how to think of  $\nabla$  as a vector of operators. One idea

$$\hat{i} \cdot \nabla = \nabla \cdot \hat{i} = \partial_x$$

to make sense of this statement apply it to a function.

§17.5 #12 / Think. Let  $f$  be scalar field and  $\vec{F}$  a vector field, ③

- a.)  $\text{curl}(f)$  is nonsense, can't take crossproduct of scalar.
- b.)  $\text{grad}(f) = \langle f_x, f_y, f_z \rangle$  is a vector field.
- c.)  $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \partial_x P + \partial_y Q + \partial_z R$ , a scalar field.
- d.)  $\text{curl}(\text{grad}(f)) = \nabla \times \nabla f = \vec{0}$ , it's the zero vector field.
- e.)  $\text{grad}(\vec{F})$  nonsense, what does  $\overset{\wedge}{\overset{\wedge}{i}}$  mean?
- f.)  $\text{grad}(\text{div}(\vec{F})) = \nabla(\nabla \cdot \vec{F})$   
 $= \langle \partial_x(\partial_x P + \partial_y Q + \partial_z R), \partial_y(\nabla \cdot \vec{F}), \partial_z(\nabla \cdot \vec{F}) \rangle$   
( $\text{grad}(\text{div}(\vec{F}))$  is a vector field)

g.)  $\text{div}(\text{grad}(f)) = \nabla \cdot \nabla f$   
 $= \nabla \cdot \langle f_x, f_y, f_z \rangle$   
 $= f_{xx} + f_{yy} + f_{zz} = \underbrace{\nabla^2 f}$   
this is a scalar field. It's known as the "Laplacian of  $f$ "

h.)  $\text{grad}(\text{div } f)$ , nonsense, can take divergence of scalar field. (what does the "dot" in  $\nabla \cdot f$  mean here?)

i.)  $\text{curl}(\text{curl } \vec{F}) = \nabla \times (\nabla \times \vec{F}) = \vec{0}$  the zero vector field

j.)  $\text{div}(\text{div } \vec{F}) = \nabla \cdot (\nabla \cdot \vec{F})$   
scalar field, but  $\nabla \cdot$  scalar field is not defined.

k.)  $\underbrace{\text{grad}(f)}_{\text{vector field}} \times \underbrace{\text{div}(\vec{F})}_{\text{scalar}} = \text{what? (Nonsense!)}$

l.)  $\text{div}(\text{curl}(\text{grad}(f))) = \nabla \cdot (\underbrace{\nabla \times \nabla f}_{\text{vector field}}) = \text{scalar field, it's zero.}$   
in this case,

Remark:  $\nabla^2 \vec{F} = (\partial_x^2 + \partial_y^2 + \partial_z^2) \langle P, Q, R \rangle = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$

§17.5 #13) Is  $\vec{F}$  conservative? That is, does  $\exists f$  with  $\vec{F} = \nabla f$ ? (4)

$$\vec{F}(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$$

You can calculate  $\nabla \times \vec{F} = 0$  thus there exists  $f$  with  $\vec{F} = \nabla f$ . ( $\text{dom } (\vec{F}) = \mathbb{R}^3$  which is simply connected thus  $\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$ ). That said, we could just as well just find  $f$ , that proves the same, anyway we're asked to find  $f$ . Need

$$\text{I.) } \frac{\partial f}{\partial x} = y^2 z^3 \longrightarrow f = \int y^2 z^3 dx + C_1(y, z) \\ = \underline{xy^2 z^3} + C_1(y, z) \quad (*)$$

$$\text{II.) } \frac{\partial f}{\partial y} = 2xyz^3$$

$$\text{III.) } \frac{\partial f}{\partial z} = 3xy^2 z^2$$

Substitute (\*) into (II.)

$$\frac{\partial f}{\partial y} = \cancel{2xyz^3} + \frac{\partial C_1}{\partial y} = \cancel{2xyz^3} \quad \therefore \frac{\partial C_1}{\partial y} = 0 \\ \Rightarrow \underline{C_1(y, z) = C_2(z)},$$

$$\text{Thus } \underline{f(x, y, z) = xy^2 z^3 + C_2(z)} \quad (**)$$

Substitute (\*\*) into (III.)

$$\frac{\partial f}{\partial z} = \cancel{3xy^2 z^2} + \frac{\partial C_2}{\partial z} = \cancel{3xy^2 z^2} \quad \therefore \frac{\partial C_2}{\partial z} = 0$$

We find  $\boxed{f(x, y, z) = xy^2 z^3 + C}$   $\Rightarrow \underline{C_2(z) = C}$ .  
(it's a constant)

gives  $\nabla f = \vec{F}$ . This works for any  $C$  we like. You could use  $C=0$  or  $C=1$  or whatever. There is a freedom to shift the potential function by a constant.

Remark: YES, we could also guess  $f(x, y, z) = xy^2 z^3$ . It's obvious.

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§ 17.5 #20 Is there a vector field  $\vec{G}$  on  $\mathbb{R}^3$  such that  $\operatorname{curl}(\vec{G}) = \langle xyz, -y^2z, yz^2 \rangle$ , It is known that  $\nabla \cdot (\nabla \times \vec{G}) = 0$ , consider

$$\begin{aligned}\nabla \cdot \langle xyz, -y^2z, yz^2 \rangle &= \partial_x(xyz) - \partial_y(y^2z) + \partial_z(yz^2) \\ &= yz - 2yz + 2yz \\ &= yz \neq 0 \quad \text{therefore no.} \\ &\quad \left( \begin{array}{l} \text{will not have} \\ \nabla \cdot (\nabla \times \vec{G}) = 0 \\ \text{as we must.} \end{array} \right)\end{aligned}$$

§ 17.5 #30 Calculate the following, Let  $\vec{r} = \langle x, y, z \rangle$  and  $|\vec{r}| = r$ ,

$$(a.) \nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3.$$

$$(b.) \nabla \cdot (r\vec{r}) = (\nabla r) \cdot \vec{r} + r(\nabla \cdot \vec{r}) \quad : \text{"product" rule.}$$

$$\begin{aligned}&= \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle \cdot \langle x, y, z \rangle + 3r \quad \text{using Lemma I} \\ &= \frac{1}{r}(x^2 + y^2 + z^2) + 3r \\ &= \frac{1}{r}(r^2) + 3r \\ &= 4r.\end{aligned}$$

Lemma I:  $\nabla r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$ .

Pf  $\frac{\partial}{\partial x}(\sqrt{x^2+y^2+z^2}) = \frac{\partial x}{\partial \sqrt{x^2+y^2+z^2}} = \frac{x}{r}$  likewise  $\frac{\partial}{\partial y}(r) = \frac{y}{r}$

and  $\frac{\partial}{\partial z}(r) = \frac{z}{r}$  thus  $\nabla r = \langle x/r, y/r, z/r \rangle$ .

Lemma II:  $\nabla r = \frac{1}{r} \vec{r}$

Pf see Lemma I,  $\nabla r = \langle x/r, y/r, z/r \rangle = \frac{1}{r} \langle x, y, z \rangle = \frac{1}{r} \vec{r}$ .

§17.5 # 30 Continued

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$$\begin{aligned}
 (c.) \quad \nabla^2 r^3 &= \frac{\partial^2}{\partial x^2}(r^3) + \frac{\partial^2}{\partial y^2}(r^3) + \frac{\partial^2}{\partial z^2}(r^3) \\
 &= \frac{\partial}{\partial x}\left(3r^2 \frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial y}\left(3r^2 \frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial z}\left(3r^2 \frac{\partial r}{\partial z}\right) \\
 &= 3\left(\frac{\partial}{\partial x}\left(r^2 \frac{x}{r}\right) + \frac{\partial}{\partial y}\left(r^2 \frac{y}{r}\right) + \frac{\partial}{\partial z}\left(r^2 \frac{z}{r}\right)\right) \\
 &= 3\left(\frac{\partial}{\partial x}(xr) + \frac{\partial}{\partial y}(ry) + \frac{\partial}{\partial z}(rz)\right) \\
 &= 3\left(r + x \frac{\partial r}{\partial x} + r + y \frac{\partial r}{\partial y} + r + z \frac{\partial r}{\partial z}\right) \\
 &= 3\left(3r + x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right) \\
 &= 3\left(3r + \frac{x^2 + y^2 + z^2}{r}\right) \\
 &= 3\left(3r + r^2/r\right) \\
 &= 3(4r) \\
 &= 12r.
 \end{aligned}$$

Remark: maybe you saw a more compact way to calculate this one. It's probably quicker to just use  $r^3 = (x^2 + y^2 + z^2)^{3/2}$ . Certainly there are many nice formulas to use, such as

$$\nabla r = \vec{r}/r$$

$$\nabla(1/r) = -\vec{r}/r^3$$

$$\nabla \times \vec{r} = 0$$

$$\nabla(\ln(r)) = \vec{r}/r^2$$

(see problem #31, you're asked to prove these)