

Homework 13, Calculus III

§17.2 #1] Let  $C$  be the path  $x = t^3$ ,  $y = t$ ,  $0 \leq t \leq 2$ .

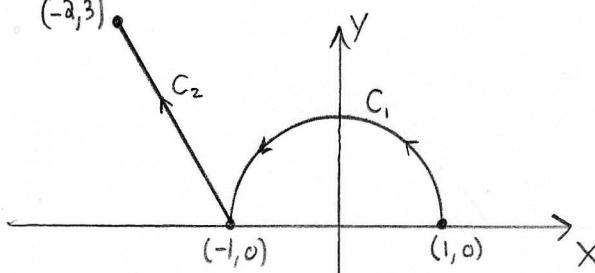
$$\vec{r}(t) = \langle t^3, t \rangle \rightarrow \frac{d\vec{r}}{dt} = \langle 3t^2, 1 \rangle \rightarrow \left| \frac{d\vec{r}}{dt} \right| = \sqrt{9t^4 + 1}$$

$$\text{Moreover, } \frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| = \sqrt{9t^4 + 1}$$

$$\begin{aligned} \int_C y^3 ds &= \int_0^2 t^3 \frac{ds}{dt} dt \\ &= \int_0^2 t^3 \sqrt{9t^4 + 1} dt \\ &= \frac{1}{36} \int_1^{145} \sqrt{u} du \\ &= \frac{1}{36} \left( \frac{2}{3} u^{3/2} \Big|_1^{145} \right) \\ &= \boxed{\frac{1}{54} \left( (145)^{3/2} - 1 \right)} \end{aligned}$$

$$\begin{array}{l} u = 9t^4 + 1 \\ du = 36t^3 dt \\ u(0) = 1 \\ u(2) = 145 \end{array} \quad \leftarrow$$

§17.2 #8] Let  $C = C_1 \cup C_2$  where  $C_1$  &  $C_2$  are pictured below. These have parametrizations,



$$\begin{aligned} C_1: \quad \vec{r}_1(\theta) &= \langle \cos \theta, \sin \theta \rangle \quad 0 \leq \theta \leq \pi \\ x &= \cos \theta, \quad y = \sin \theta \\ dx &= -\sin \theta d\theta \quad dy = \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} C_2: \quad \vec{r}_2(t) &= (-1,0)(1-t) + t(-2,3) \\ &= \langle -1+t-3t, 3t \rangle \\ x &= -t-1 \quad y = 3t \\ dx &= -dt \quad dy = 3dt \end{aligned}$$

Remark: I write  $\vec{r}_1(\theta)$  and  $\vec{r}_2(t)$  for fun, we really just use  $x, y, dx, dy$  in the calculation below. There are many ways to organize work here.

Remark: Use standard trick. The parametrized line from  $P$  to  $Q$  is simply  $\vec{r}(t) = P(1-t) + tQ$ ,  $0 \leq t \leq 1$ . You can check  $\vec{r}(0) = P$ ,  $\vec{r}(1) = Q$ . Also  $\vec{r}(t) = P + t \underbrace{(Q-P)}_{\vec{v} \text{ from before.}}$

§17.2 #8 continued

(2)

$$\begin{aligned}
 \int_{C_1} \sin(x)dx + \cos(y)dy &= \int_0^\pi -(\sin(\cos\theta)\sin\theta d\theta + \cos(\sin\theta)\cos\theta d\theta) \\
 &= \int_0^\pi \sin(\cos\theta)(-\sin\theta dt) + \int_0^\pi \cos(\sin\theta)\cos\theta d\theta \\
 &= \int_1^1 \sin(u)du + \int_0^0 \cancel{\cos(w)dw} \quad \begin{array}{l} u = \cos\theta \\ du = -\sin\theta d\theta \\ w = \sin\theta \\ dw = \cos\theta d\theta \end{array} \\
 &= -\cos(u) \Big|_1^{-1} \\
 &= \cos(1) - \cos(-1) \\
 &= \boxed{0}
 \end{aligned}$$

$$\begin{aligned}
 \int_{C_2} \sin(x)dx + \cos(y)dy &= \int_0^1 \sin(-t-1)(-dt) + \cos(3t)3dt \\
 &= \int_0^1 \sin(t+1)dt + \int_0^1 3\cos(3t)dt \\
 &= -\cos(t+1) \Big|_0^1 + \sin(3t) \Big|_0^1 \\
 &= -\cos(2) + \cos(1) + \sin(3) - \sin(0) \\
 &= \boxed{\sin(3) - \cos(2) + \cos(1)}
 \end{aligned}$$

$$\begin{aligned}
 \int_C \sin(x)dx + \cos(y)dy &= \int_{C_1} \sin(x)dx + \cos(y)dy + \int_{C_2} \sin(x)dx + \cos(y)dy \\
 &= \boxed{\sin(3) - \cos(2) + \cos(1)}
 \end{aligned}$$

§17.2 #11] C is line segment from  $(0, 0, 0)$  to  $(1, 2, 3)$ . Thus (3)

$$\vec{r}(t) = P(1-t) + tQ = t(1, 2, 3) = \langle t, 2t, 3t \rangle \quad 0 \leq t \leq 1$$

$$\vec{r}'(t) = \langle 1, 2, 3 \rangle$$

$$|\vec{r}'(t)| = \sqrt{1+4+9} = \sqrt{14}$$

$$ds = \frac{ds}{dt} dt = |\vec{r}'(t)| dt = \sqrt{14} dt$$

Thus,

$$\begin{aligned}\int_C xe^{yz} ds &= \int_0^1 te^{6t^2} \sqrt{14} dt \\ &= \left. \frac{\sqrt{14}}{12} e^{6t^2} \right|_0^1 \\ &= \boxed{\frac{\sqrt{14}}{12} (e^6 - 1)}\end{aligned}$$

§17.2 #19] Find  $\int_C \vec{F} \cdot d\vec{r}$  with  $C: \vec{r}(t) = \langle 11t^4, t^3 \rangle$  for  $0 \leq t \leq 1$ ,  
and  $\vec{F}(x, y) = \langle xy, 3y^2 \rangle$ .

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_0^1 \langle 11t^7, 3t^6 \rangle \cdot \langle 44t^3, 3t^2 \rangle dt \\ &= \int_0^1 (11(44)t^{10} + 9t^8) dt \\ &= \left. \left( \frac{11(44)t^9}{11} + \frac{9t^9}{9} \right) \right|_0^1 \\ &= 44 + 1 \\ &= \boxed{45}.\end{aligned}$$

§17.2 #2a Calculate  $\int_C \vec{F} \cdot d\vec{r}$  for  $\vec{F}(x, y, z) = \langle z, y, -x \rangle$  with the curve  $C$  parameterized by  $\vec{r}(t) = \langle t, \sin t, \cos t \rangle$ ,  $0 \leq t \leq \pi$ . (4)

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^\pi \langle \cos t, \sin t, -t \rangle \cdot \langle 1, \cos t, -\sin t \rangle dt \\ &= \int_0^\pi (\cos t + \sin t \cos t + t \sin t) dt \quad \leftarrow \text{using } (*) \\ &= \left( \sin t + \frac{1}{2} \sin^2 t - t \cos t + \sin t \right) \Big|_0^\pi \\ &= -\pi \cos(\pi), \text{ other terms are zero.} \\ &= \boxed{\pi}\end{aligned}$$

(\*) The integrals in this problem follow from u-substitution & Integration By Parts

$$\int \underbrace{\sin t \cos t dt}_{u \ du} = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} \sin^2 t + C.$$

$$\int \underbrace{t \sin t dt}_{u \ dv} = uv - \int v du = -t \cos(t) + \int \cos(t) dt = -t \cos t + \sin t + C.$$

§17.2 #41 Find work done by  $\vec{F}(x, y, z) = \langle y+z, x+z, x+y \rangle$  on a particle moving along line segment from  $(1, 0, 0)$  to  $(3, 4, 2)$ .

Notice  $f(x, y, z) = xy + xz + yz$  has  $\vec{F} = \nabla f$ . We can use the FTC for line integrals, let  $C$  be the line segment

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(3, 4, 2) - f(1, 0, 0) \\ &= 3(4) + 3(2) + 4(2) - 1(0) - 1(0) - 0(0) \\ &= 12 + 6 + 8 \\ &= \boxed{26}.\end{aligned}$$

Remark: if you wish to calculate w/o FTC can use  $\vec{r}(t) = P(1-t) + tQ = (1, 0, 0)(1-t) + t(3, 4, 2)$  which yields  $x = 1+2t$ ,  $y = 4t$ ,  $z = 2t$ ,  $0 \leq t \leq 1$ .

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (4t+2t) 2dt + (1+2t+2t) 4dt + (1+2t+4t) 2dt = \int_0^1 (40t+6) dt = \boxed{26}.$$

Homework 14, Calculus III

§17.3 #3 Is  $\vec{F}(x,y) = \langle 2x - 3y, -3x + 4y - 8 \rangle$  conservative?

We wish to find  $f$  such that  $\nabla f = \vec{F}$ . This requires

$$f_x = 2x - 3y \Rightarrow f(x,y) = x^2 - 3xy + C_1(y)$$

$$f_y = -3x + 4y - 8 \Rightarrow f(x,y) = -3xy + 2y^2 - 8y + C_2(x)$$

We can see that  $C_1(y) = 2y^2 - 8y$  while  $C_2(x) = x^2$  hence

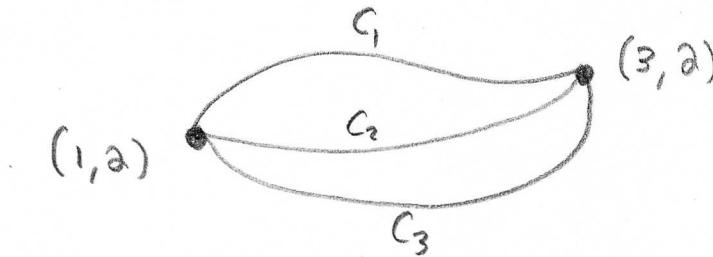
$$\boxed{f(x,y) = x^2 - 3xy + 2y^2 - 8y}$$

So yes  $\vec{F}$  is conservative,  $\vec{F} = \nabla f$  for the  $f$  just constructed.

Remark: the technique I just used to find  $f$  is equivalent to the one used in my notes. However, sometimes it's more efficient to do multiple integrations as opposed to integrate, substitute, integrate... You find  $f$  however seems most convenient. Just find a way.

§17.3 #11 We have a figure showing  $\vec{F}(x,y) = \langle 2xy, x^2 \rangle$  and three curves from  $(1,2)$  to  $(3,2)$ . Why is

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r} ?$$



Notice that  $f(x,y) = x^2y$  gives  $\vec{F} = \nabla f$ . Thus the FTC for line integrals answers the question.

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r} = f(3,2) - f(1,2) \\ &= (3)^2 2 - (1)^2 2 \\ &= \boxed{16} \end{aligned}$$

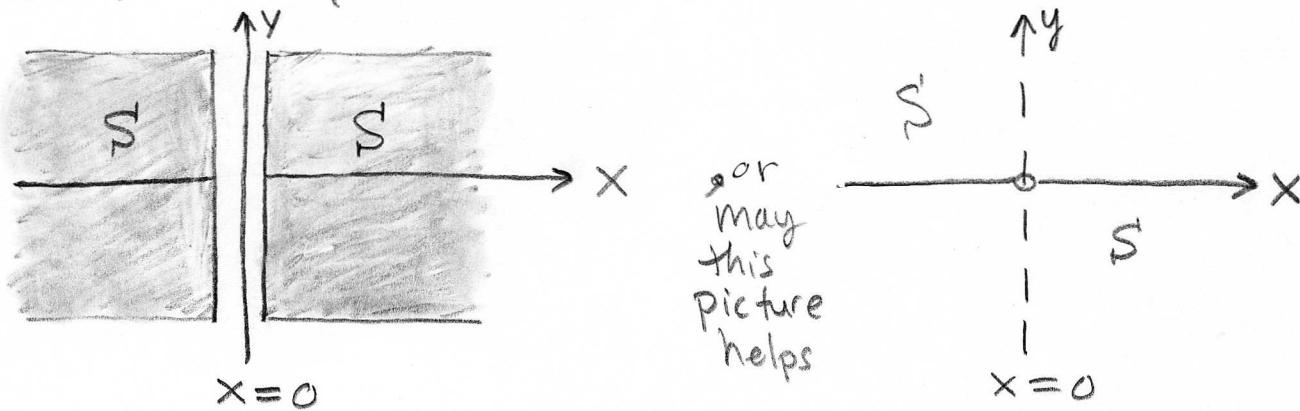
§17.3 #21 Find work done by  $\vec{F}(x,y) = \langle 2y^{3/2}, 3x\sqrt{y} \rangle$  in moving an object from  $P = (1,1)$  to  $Q = (2,4)$ . Notice no path is specified,  $\vec{F}$  had better be conservative (hence path-independent) or else this problem is crazy.

$$f(x,y) = 2xy^{3/2} \text{ has } \vec{F} = \nabla f$$

Then we can use FTC,

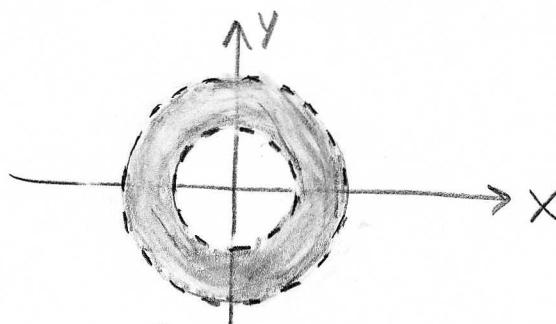
$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,4) - f(1,1) \\ &= 2(2)4^{3/2} - 2(1)(1) \\ &= 4(8) - 2 \\ &= \boxed{30}. \end{aligned}$$

§17.3 #30 Is this set open, connected, and/or simply connected?  
Consider  $S' = \{ (x,y) \mid x \neq 0 \}$



This set is open, not connected and not simply connected. The sets  $x > 0$  and  $x < 0$  are themselves both connected & simply connected.

§17.3 #31 Consider  $S' = \{ (x,y) \mid 1 < x^2 + y^2 < 4 \}$



This set is open and connected.  
This set is not simply connected, it has a hole on which loops will get caught.

Remarks: if you really want to delve deeper read a book on topology.