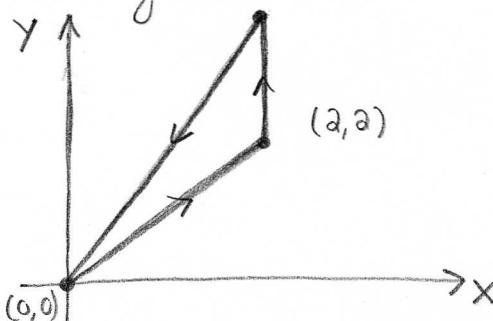


Homework 15, Calculus III

(1)

§17.4 #5] Use Green's Theorem to evaluate line integral along the given positively oriented curve. Let

$C$  be triangle with vertices  $(0,0), (2,2), (2,4)$



top line is  $y = 2x$

bottom line is  $y = x$ .

point in triangle  $S$  has

$$0 \leq x \leq 2, \quad x \leq y \leq 2x.$$

Use Green's Theorem

$$\int_C \underbrace{xy^2 dx}_P + \underbrace{2x^2 y dy}_Q = \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (\partial S = C)$$

↑  
boundary  
of  $S$

$$= \int_0^2 \int_x^{2x} (4xy - 2xy) dy dx$$

$$= \int_0^2 \int_x^{2x} 2xy dy dx$$

$$= \int_0^2 \left( xy^2 \Big|_x^{2x} \right) dx$$

$$= \int_0^2 (4x^3 - x^3) dx$$

$$= \frac{3x^4}{4} \Big|_0^2$$

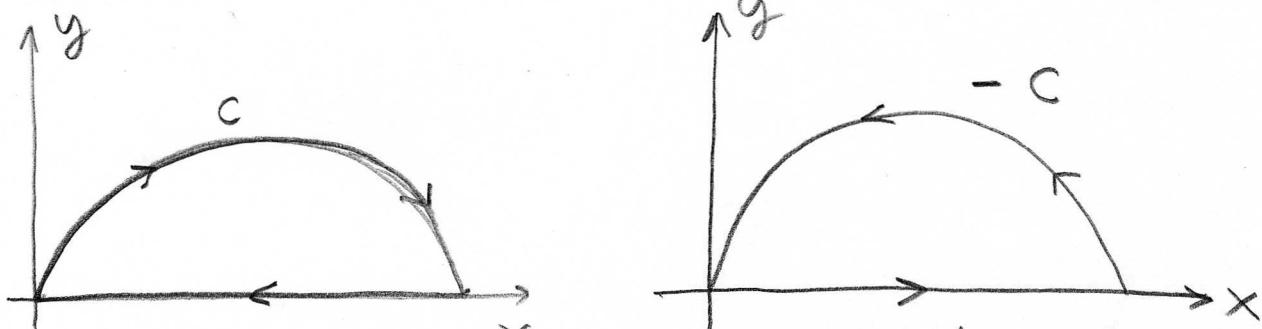
$$= \frac{3}{4} (16)$$

$$= \boxed{12}$$

We see that Green's Theorem converted a line integral into an associate double integral.

(2)

§17.4 #11) Let  $\vec{F}(x, y) = \langle -\sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$  calculate  
 $\int_C \vec{F} \cdot d\vec{r}$  over  $C$  which is  $y = \sin(x)$  from  $(0, 0)$  to  $(\pi, 0)$   
and then the line segment from  $(\pi, 0)$  to  $(0, 0)$ .



the given curve has negative orientation. We must apply Green's Th<sup>mo</sup> to  $-C$  instead.

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \int_0^\pi \int_0^{\sin(x)} (-2x - 3y^2) dy dx \\
&= \int_0^\pi (2x \sin(x) - \sin^3(x)) dx \\
&= \int_0^\pi \underbrace{2x \sin(x) dx}_u - \int_0^\pi \underbrace{(1 - \cos^2(x)) \sin(x) dx}_{1-u^2} - du \\
&= -2x \cos(x) \Big|_0^\pi + \int_0^\pi 2 \cos(x) dx + \int_1^{-1} (1 - u^2) du \\
&= -2\pi \cos(\pi) + 2 \sin(x) \Big|_0^\pi + \left( u - \frac{1}{3} u^3 \right) \Big|_1^{-1} \\
&= 2\pi + \left( -1 + \frac{1}{3} \right) - \left( 1 - \frac{1}{3} \right) \\
&= 2\pi - 2 + \frac{2}{3} \\
&= 2\pi - \frac{4}{3}
\end{aligned}$$

$u(\pi) = -1$   
 $u(0) = 1$

Thus  $\int_C \vec{F} \cdot d\vec{r} = - \int_{-C} \vec{F} \cdot d\vec{r} = \boxed{\frac{4}{3} - 2\pi}$

§17.6 #37 Find the area of the part of the plane  $3x + 2y + z = 6$  ③ that lies in the first octant. We must find the parametrization to begin with,

$$\vec{r}(x, y) = \langle x, y, 6 - 3x - 2y \rangle \quad \text{for } \underbrace{6 - 3x - 2y \geq 0}_{\text{let's think about this.}}$$

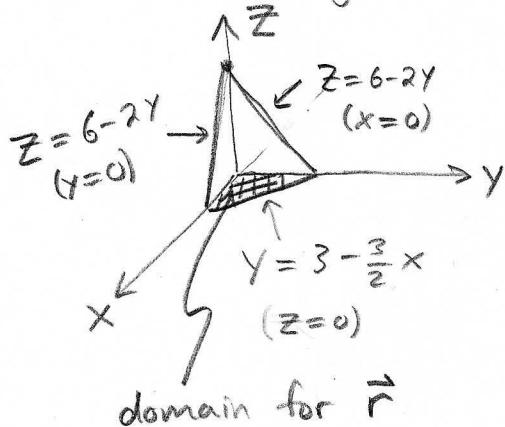
$$6 - 3x - 2y \geq 0$$

$$-3 + \frac{3}{2}x + y \leq 0$$

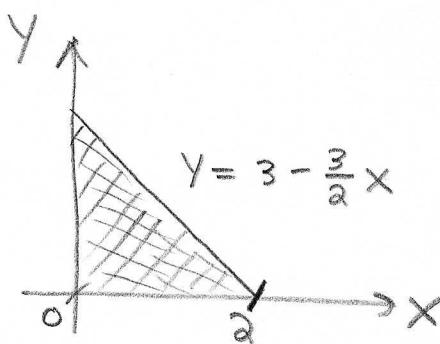
$$-3 + \frac{3}{2}x \leq -y$$

$$3 - \frac{3}{2}x \geq y \Rightarrow 0 \leq y \leq 3 - \frac{3}{2}x$$

We need to figure out the domain for  $\vec{r}(x, y)$ .



$$\begin{aligned} Z &= 6 - 3x - 2y \\ Z &= 0, \quad 6 - 3x - 2y = 0 \\ 2y &= 6 - 3x \\ Y &= 3 - \frac{3}{2}x \\ Y &= 0, \quad Z = 6 - 3x \\ X &= 0, \quad Z = 6 - 2y \end{aligned}$$



$$\text{dom}(\vec{r}) = \left\{ (x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3 - \frac{3}{2}x \right\}$$

Philosophy: keep drawing until it becomes clear. Maybe you don't need to draw as much as me here.

§ 17.6 #37 Continued

(4)

The surface is described by  $\vec{r}(x,y) = \langle x, y, 6 - 3x - 2y \rangle$   
 for  $(x,y) \in \text{dom}(\vec{r})$  given on last page, observe

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, -3 \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, -2 \rangle$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \langle 3, 2, 1 \rangle = \vec{N}(x,y)$$

The normal vector field is constant for this rather  
 special surface  $3x + 2y + z = 6$ . (No SURPRISE !)

$$\begin{aligned} A &= \iint_{\text{dom}(\vec{r})} |\vec{N}(x,y)| dA = \int_0^2 \int_0^{3 - \frac{3x}{2}} \sqrt{9+4+1} dy dx \\ &= \int_0^2 \sqrt{14} \left( 3 - \frac{3x}{2} \right) dx \\ &= \sqrt{14} \left( 3x - \frac{3x^2}{4} \right) \Big|_0^2 \\ &= \sqrt{14} (6 - 3) \\ &= \boxed{3\sqrt{14}} \end{aligned}$$

5

§17.6 #41 Find the area of the surface  $z = xy$  that lies with the cylinder  $x^2 + y^2 = 1$ . This surface is described by the parametrization,

$$\vec{r}(x, y) = \langle x, y, xy \rangle \text{ with } x^2 + y^2 \leq 1$$

We need to find the normal vector field,

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, y \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, x \rangle$$

$$\vec{N}(x, y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \langle -y, -x, 1 \rangle$$

$$|\vec{N}(x, y)| = \sqrt{y^2 + x^2 + 1}$$

$$dA = |\vec{N}(x, y)| dx dy = \sqrt{1+x^2+y^2} dx dy$$

Hence, the surface area is

$$\begin{aligned}
 A &= \iint_{x^2+y^2 \leq 1} \sqrt{1+x^2+y^2} dx dy \quad \rightarrow \text{polar change of variables makes integration easy} \\
 &= \int_0^{2\pi} \int_0^1 \sqrt{1+r^2} r dr d\theta \quad \leftarrow \\
 &= \int_0^{2\pi} d\theta \int_1^2 \frac{1}{2} \sqrt{u} du \quad \left( \begin{array}{ll} u = 1+r^2 & u(0) = 1 \\ du = 2r dr & u(1) = 2 \end{array} \right) \\
 &= (2\pi) \frac{1}{2} \frac{2}{3} u^{3/2} \Big|_1^2 \\
 &= \frac{2\pi}{3} (2^{3/2} - 1) \\
 &= \boxed{\frac{2\pi}{3} (2\sqrt{2} - 1)}
 \end{aligned}$$