

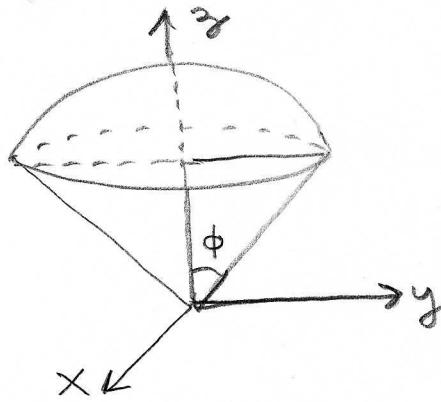
§17.7 #5 | Let  $S$  be the part of the plane  $z = 1 + 2x + 3y$  that lies above the rectangle  $[0, 3] \times [0, 2]$ . We can calculate  $\vec{r}(x, y) = \langle x, y, 1+2x+3y \rangle$  has normal vector field  $\vec{N}(x, y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = \langle -2, -3, 1 \rangle$  hence  $dS = |\vec{N}(x, y)| dx dy = \sqrt{14} dx dy$ .

$$\begin{aligned}
 \iint_S x^2 y z dS &= \int_0^3 \int_0^2 x^2 y (1+2x+3y) \sqrt{14} dy dx \\
 &= \int_0^3 \int_0^2 (x^2 y + 2x^3 y + 3x^2 y^2) \sqrt{14} dy dx \\
 &= \int_0^3 \left\{ \left( x^2 + 2x^3 \right) \frac{y^2}{2} \Big|_0^2 + x^2 y^3 \Big|_0^2 \right\} \sqrt{14} dx \\
 &= \int_0^3 \left[ \left( x^2 + 2x^3 \right) \frac{4}{2} + 8x^2 \right] \sqrt{14} dx \\
 &= \int_0^3 (10x^2 + 4x^3) \sqrt{14} dx \\
 &= \left[ \left( \frac{10x^3}{3} + x^4 \right) \Big|_0^3 \right] \sqrt{14} \\
 &= (90 + 81) \sqrt{14} \\
 &= \boxed{171 \sqrt{14}}
 \end{aligned}$$

§ 17.7 #14 | Calculate  $\iint_S y^2 dS$  for  $S$  the part of the sphere  $x^2 + y^2 + z^2 = 4$  that lies inside the cylinder  $x^2 + y^2 = 1$  above the  $xy$ -plane ②

$$\Sigma(\theta, \phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi \rangle$$

The question is what is the domain of  $\Sigma$ ?



$$\sin\phi = \frac{1}{2}$$

$$\Rightarrow \phi = \pi/6$$

where the sphere intersects the cylinder.

We find  $0 \leq \theta \leq 2\pi$  and  $0 \leq \phi \leq \pi/6$ .

$$\frac{\partial \Sigma}{\partial \theta} = \langle -2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0 \rangle$$

$$\frac{\partial \Sigma}{\partial \phi} = \langle 2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi \rangle$$

$$\vec{N}(\theta, \phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix}$$

$$= \langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin^2\theta\sin\phi\cos\phi - 4\cos^2\theta\sin\phi\cos\phi \rangle$$

$$= -4 \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi \rangle$$

The magnitude of the normal vector field,

$$|\vec{N}(\theta, \phi)| = 4\sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= 4\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= 4\sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)}$$

$$= 4\sqrt{\sin^2\phi}$$

$$= 4\sin\phi \quad \longrightarrow \quad dS = 4\sin\phi d\theta d\phi$$

§17.7 #14

(3)

$$\begin{aligned}
 \iint_S y^2 dS &= \int_0^{2\pi} \int_0^{\pi/6} (2\sin\theta\sin\phi)^2 4\sin\phi d\phi d\theta \\
 &= \int_0^{2\pi} \sin^2\theta d\theta \int_0^{\pi/6} 16\sin^3\phi d\phi \\
 &= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta \int_0^{\pi/6} 16(1-\cos^2\phi)\sin\phi d\phi \\
 &= \left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right) \Big|_0^{2\pi} \left(16 \left[\frac{\cos^3\phi}{3} - \cos\phi\right]\right) \Big|_0^{\pi/6} \\
 &= \left(\frac{2\pi}{2}\right) \left(16 \left[\frac{\cos^3(\pi/6)}{3} - \cos(\pi/6) - \frac{\cos^3(0)}{3} + \cos(0)\right]\right) \\
 &= 16\pi \left(\frac{1}{3} \left(\frac{\sqrt{3}}{2}\right)^3 - \frac{\sqrt{3}}{2} - \frac{1}{3} + 1\right) \\
 &= 16\pi \left(\frac{3\sqrt{3}}{3(8)} - \frac{\sqrt{3}}{2} + \frac{2}{3}\right) \\
 &= \boxed{16\pi \left(\frac{2}{3} - \frac{3\sqrt{3}}{8}\right)} = \boxed{\pi \left(\frac{32}{3} - 6\sqrt{3}\right)}
 \end{aligned}$$

§17.7 #18) Calculate  $\iint_S (x^2 + y^2 + z^2) dS$  where  $S$  is the part of the cylinder  $x^2 + y^2 = 9$  between the planes  $z=0$  &  $z=2$ . Parametrize with help of cylindrical coordinates. We want to have  $r^2 = 9$  so  $r=3$  while  $0 \leq z \leq 2$  &  $0 \leq \theta \leq 2\pi$

$$\begin{aligned}
 x &= r\cos\theta = 3\cos\theta \\
 y &= r\sin\theta = 3\sin\theta \\
 z &= z
 \end{aligned}$$

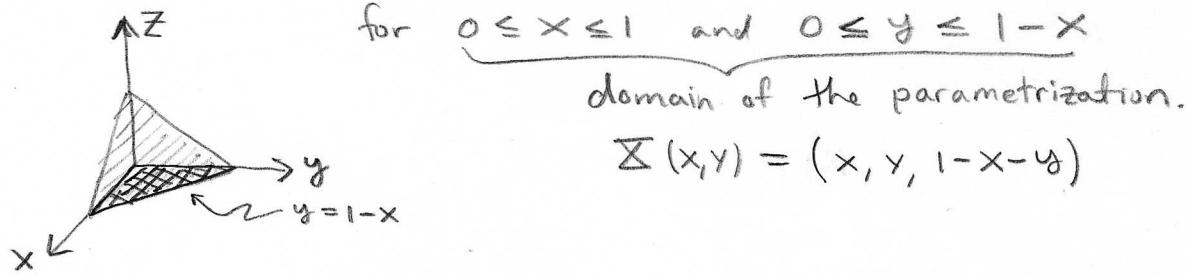
Hence  $\Sigma(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle$  and recall that  $dS = |\vec{dS}| = |\Sigma_\theta \times \Sigma_z d\theta dz| = |3\vec{e}_r d\theta dz| = 3d\theta dz$ .

$$\left( \frac{\partial \Sigma}{\partial \theta} \times \frac{\partial \Sigma}{\partial z} \right) = \begin{vmatrix} i & j & k \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 3\cos\theta, 3\sin\theta, 0 \rangle = 3\vec{e}_r$$

§17.7 #18 We found  $dS = 3d\theta dz$  with  $x = 3\cos\theta$ ,  $y = 3\sin\theta$ ,  $z = z$  for  $0 \leq \theta \leq 2\pi$  and  $0 \leq z \leq 2$  describe  $S$  parametrically.

$$\begin{aligned}
 \iint_S x^2 + y^2 + z^2 dS &= \int_0^2 \int_0^{2\pi} (9 + z^2) 3d\theta dz \\
 &= \int_0^2 (27 + 3z^2) \theta \Big|_0^{2\pi} dz \\
 &= \int_0^2 (54\pi + 6\pi z^2) dz \\
 &= 54\pi z \Big|_0^2 + 2\pi z^3 \Big|_0^2 \\
 &= 108\pi + 16\pi \\
 &= \boxed{124\pi} \quad \text{add top/bottom (see 46 2)}
 \end{aligned}$$

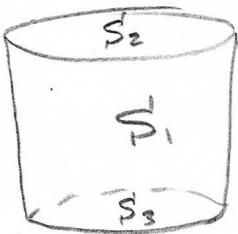
§17.7 #21 Calculate the surface integral  $\iint_S \vec{F} \cdot d\vec{S}$  for  $\vec{F} = \langle xze^y, -xze^y, z \rangle$  and  $S$  the part of the plane  $x+y+z=1$  in the first octant with downward orientation. It's fairly clear that  $d\vec{S} = -\langle 1, 1, 1 \rangle dx dy$  if we choose  $x$  and  $y$  as parameters. We have  $x=x$ ,  $y=y$ ,  $z=1-x-y$



$$\begin{aligned}
 \iint_S \vec{F} \cdot d\vec{S} &= \iint_{\text{dom}(\Sigma)} \vec{F}(\Sigma(x, y)) \cdot \langle -1, -1, -1 \rangle dx dy \\
 &= - \int_0^1 \int_0^{1-x} \langle x(1-x-y)e^y, -x(1-x-y)e^y, 1-x-y \rangle \cdot \langle 1, 1, 1 \rangle dy dx \\
 &= - \int_0^1 \int_0^{1-x} (x(1-x-y)e^y - x(1-x-y)e^y + 1-x-y) dy dx \\
 &= \int_0^1 \int_0^{1-x} (y+x-1) dy dx \\
 &= \int_0^1 \left( \frac{1}{2}(1-x)^2 + (x-1)(1-x) \right) dx \\
 &= \int_0^1 \left( \frac{1}{2}(1-x)^2 - (1-x)^2 \right) dx = \int_0^1 -\frac{1}{2}(1-x)^2 dx = \frac{1}{6}(1-x)^3 \Big|_0^1 \\
 &= \boxed{-\frac{1}{6}}
 \end{aligned}$$

S 17.7 #18 Continued

(oops, I forgot the ends of the cylinder my first time through. We already found the curved part contributes  $124\pi$  to  $\iint_S (x^2+y^2+z^2) dS$ .



$$S_2: \mathbf{X}_2(x, y) = \langle x, y, 2 \rangle$$

for  $x^2+y^2 \leq 3$ . Calculate the normal

$$\vec{N}_2(x, y) = \frac{\partial \mathbf{X}_2}{\partial x} \times \frac{\partial \mathbf{X}_2}{\partial y} = \hat{i} \times \hat{j} = \hat{k}$$

thus,  $d\vec{S} = dx dy \hat{k}$  &  $dS = dx dy$ .

$$S_3: \mathbf{X}_3(y, x) = \langle x, y, 0 \rangle$$

for  $x^2+y^2 \leq 3$ . Calculate the normal

$$\vec{N}_3(y, x) = \frac{\partial \mathbf{X}_3}{\partial y} \times \frac{\partial \mathbf{X}_3}{\partial x} = \hat{j} \times \hat{i} = -\hat{k}$$

thus,  $d\vec{S} = -dx dy \hat{k}$  &  $dS = dx dy$

Not surprisingly we found  $dS = dA$  for  $S_2$  and  $S_3$  because the caps are flat and parallel to the  $xy$ -plane where  $dA = dx dy$  lives.

$$\begin{aligned} \iint_{S_2} (x^2+y^2+z^2) dS &= \iint_{x^2+y^2 \leq 3} (x^2+y^2+4) dx dy \\ &= \int_0^3 \int_0^{2\pi} (r^2 + 4) r dr d\theta \\ &= 2\pi \int_0^3 (r^3 + 4r) dr \\ &= 2\pi \left( \frac{81}{4} + \frac{36}{2} \right) = 2\pi \left( \frac{81+72}{4} \right) = \underline{\underline{\frac{153\pi}{2}}} \end{aligned}$$

$$\begin{aligned} \iint_{S_3} (x^2+y^2+z^2) dS &= \int_0^3 \int_0^{2\pi} r^2 r dr d\theta \\ &= (2\pi) \left( \frac{81}{4} \right) = \underline{\underline{\frac{81\pi}{2}}} \end{aligned}$$

In total then  $\iint_S (x^2+y^2+z^2) dS = 124\pi + \frac{81\pi}{2} + \frac{153\pi}{2} = \boxed{241\pi}$

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§17.7 #23 Let  $\vec{F} = \langle x, -z, y \rangle$  calculate  $\iint_S \vec{F} \cdot d\vec{S}$  over  $S$  the part of the sphere  $x^2 + y^2 + z^2 = 4$  in the first octant with orientation toward the origin.  $S$  is described by

$$\Sigma(\theta, \phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi \rangle$$

$$d\vec{S} = -\rho^2 \sin\phi e_p d\theta d\phi \leftarrow (\text{Geometry})$$

$$= -4 \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi \rangle \quad (\text{See #14 or notes})$$

with  $0 \leq \theta \leq \pi/2$  and  $0 \leq \phi \leq \pi/2$  to select 1st octant.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \int_0^{\pi/2} \int_0^{\pi/2} \langle 2\cos\theta\sin\phi, -2\cos\phi, 2\sin\theta\sin\phi \rangle \cdot \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi \rangle (-4d\theta d\phi) \\ &= \int_0^{\pi/2} \int_0^{\pi/2} -4(2\cos^2\theta\sin^3\phi - 2\sin\theta\sin^2\phi\cos\phi + 2\sin\theta\sin^2\phi\cos\phi) d\theta d\phi \\ &= -8 \int_0^{\pi/2} \cos^2\theta d\theta \int_0^{\pi/2} \sin^3\phi d\phi \\ &= -4 \int_0^{\pi/2} (1 + \cos(2\theta)) d\theta \int_0^{\pi/2} (1 - \cos^2\phi) \sin\phi d\phi \\ &= -4 \left( \theta + \frac{1}{2}\sin(2\theta) \right) \Big|_0^{\pi/2} \left( \frac{1}{3}\cos^3\phi - \cos\phi \right) \Big|_0^{\pi/2} \\ &= -4 \left( \frac{\pi}{2} \right) \left( \frac{-1}{3}\cos^3(0) + \cos(0) \right) \\ &= -2\pi (2/3) \\ &= \boxed{-4\pi/3} \end{aligned}$$

§17.7 #41] A fluid has density  $870 \text{ kg/m}^3$  and flows with velocity field  $\vec{V} = \langle z, y^2, x^2 \rangle$  where  $x, y, z$  are measured in meters and  $\vec{V}$  has components in m/s. Find the flow-rate out of the cylinder  $x^2 + y^2 = 4$  for  $0 \leq z \leq 1$ . This cylinder is conveniently described by cylindrical coordinates,

$$\Sigma_r(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle \quad (r = 2)$$

for  $0 \leq \theta \leq 2\pi$  and  $0 \leq z \leq 1$ .

$$\Sigma_\theta = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$$

$$\Sigma_z = \langle 0, 0, 1 \rangle$$

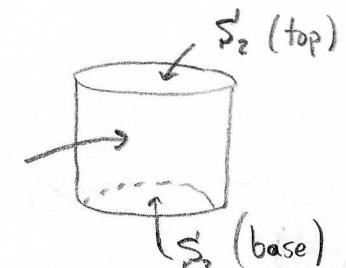
$$\Sigma_\theta \times \Sigma_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle = 2\vec{e}_r$$

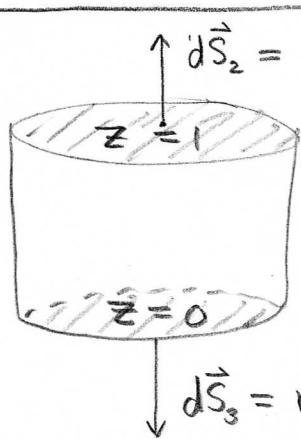
$$d\vec{S} = (\Sigma_\theta \times \Sigma_z) d\theta dz$$

Then the flux integral of  $\vec{V}$  through  $S$  gives the sum of  $[\vec{V} \cdot d\vec{S}] \leftarrow \text{volume/second}$  that flows through  $d\vec{S}$ . Notice that  $\rho = 870 \frac{\text{kg}}{\text{m}^3} = \frac{\text{mass}}{\text{volume}} = \frac{dm}{dV}$  implies  $dm = \rho dV$  thus  $\frac{dm}{dt} = \rho \frac{dV}{dt} = \rho \vec{V} \cdot d\vec{S}$

$$\begin{aligned} \iint_S \vec{V} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^1 \langle z, 4\sin^2\theta, 4\cos^2\theta \rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0 \rangle dz d\theta \\ &= \int_0^{2\pi} \int_0^1 (2z\cos\theta + 8\sin^3\theta) dz d\theta : \int_0^1 2z dz = z^2 \Big|_0^1 = 1. \\ &= \int_0^{2\pi} (\cos\theta + 8(1-\cos^2\theta)\sin\theta) d\theta \\ &= (\sin\theta + 8 \frac{\cos^3\theta}{3} - 8\cos\theta) \Big|_0^{2\pi} \\ &= 8 \left( \frac{1}{3} - \frac{1}{3} \right) = \boxed{0} \end{aligned}$$

$\uparrow$   
0 kg/s flows out of  $S$ .





$$\vec{x}_2(\theta, r) = \langle r\cos\theta, r\sin\theta, 1 \rangle$$

$$0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$$

$$\vec{x}_3(\theta, r) = \langle r\cos\theta, r\sin\theta, 0 \rangle$$

$$0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$$

$$\begin{aligned} \iint_{S_2} \langle z, y^2, x^2 \rangle \cdot d\vec{S}_2 &= \int_0^{2\pi} \int_0^2 \langle 1, r^2 \sin^2\theta, r^2 \cos^2\theta \rangle \cdot (r dr d\theta \hat{k}) \\ &= \int_0^{2\pi} \int_0^2 r^3 \cos^2\theta dr d\theta \\ &= \int_0^{2\pi} \cos^2\theta d\theta \int_0^2 r^3 dr = \underline{\underline{4\pi}} \end{aligned}$$

$$\begin{aligned} \iint_{S_3} \langle z, y^2, x^2 \rangle \cdot d\vec{S}_3 &= \int_0^{2\pi} \int_0^2 \langle 0, r^2 \sin^2\theta, r^2 \cos^2\theta \rangle \cdot (-r dr d\theta \hat{k}) \\ &= - \int_0^{2\pi} \int_0^2 r^3 \cos^2\theta dr d\theta \\ &= \underline{\underline{-4\pi}} \end{aligned}$$

Thus  $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S} = 0$  which shows that there is no net flow out of the cylinder. What has happened here is that the flow into one side is balanced out by the flow out of another side.

Remark: Can check via Divergence Thm,  $\nabla \cdot \vec{V} = 2y$  hence

$$\iint_S \vec{V} \cdot d\vec{S} = \iiint_E 2y dV = 0 \quad \text{by symmetry } y < 0 \text{ & } y > 0 \text{ equally much on } x^2 + y^2 = 4, 0 \leq z \leq 1.$$