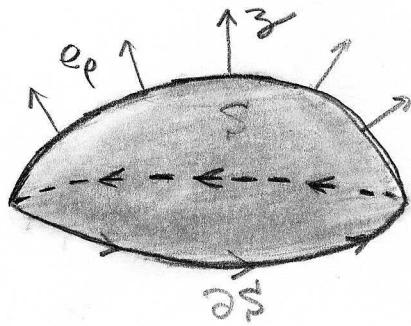


Homework 17, Calculus III

①

§ 17.8 #2] Use Stoke's Th^m to evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ for
 $\vec{F} = \langle 2y \cos(z), e^x \sin(z), xe^y \rangle$ for S the hemisphere
 $x^2 + y^2 + z^2 = 9, z \geq 0$ oriented upwards.



S has unit normal e_p

∂S has unit tangent e_θ

$$\partial S: x = 3 \cos \theta, y = 3 \sin \theta, z = 0 \\ 0 \leq \theta \leq 2\pi$$

(this is the equator)

Note $d\vec{r} = (3d\theta)e_\theta$

Use Stoke's Th^m

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle 2y \cos(z), e^x \sin(z), xe^y \rangle \cdot (3e_\theta) d\theta : \begin{matrix} x = 3 \cos \theta \\ y = 3 \sin \theta \\ z = 0 \end{matrix}$$

$$= \int_0^{2\pi} 3 \langle 6 \sin \theta, 0, 3 \cos \theta e^{3 \sin \theta} \rangle \cdot \langle \sin \theta, \cos \theta, 0 \rangle d\theta$$

$$= -18 \int_0^{2\pi} \sin^2 \theta d\theta$$

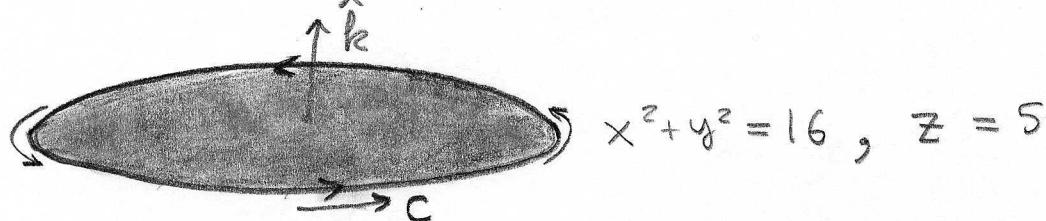
$$= -9 \int_0^{2\pi} (1 - \cos(2\theta)) d\theta$$

$$= \boxed{-9\pi}$$

§17.8 #9 Use Stoke's Th^m to calculate $\int_C \vec{F} \cdot d\vec{r}$ in the case ②
 $\vec{F}(x, y, z) = \langle yz, 2xz, e^{xy} \rangle$ and C is the circle $x^2 + y^2 = 16$
and $z = 5$ which is oriented counterclockwise from above.

$$\begin{aligned}\nabla \times \vec{F} &= \left\langle \frac{\partial}{\partial y}(e^{xy}) - \frac{\partial}{\partial z}(2xz), \frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(e^{xy}), \frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial y}(yz) \right\rangle \\ &= \langle xe^{xy} - 2x, y - ye^{xy}, 2z - z \rangle \\ &= \langle x(e^{xy} - 2), y(1 - e^{xy}), z \rangle\end{aligned}$$

Let's picture this circle, it is the boundary of the disk S ; $\partial S = C$ and S has $d\vec{s} = r dr d\theta \hat{k}$



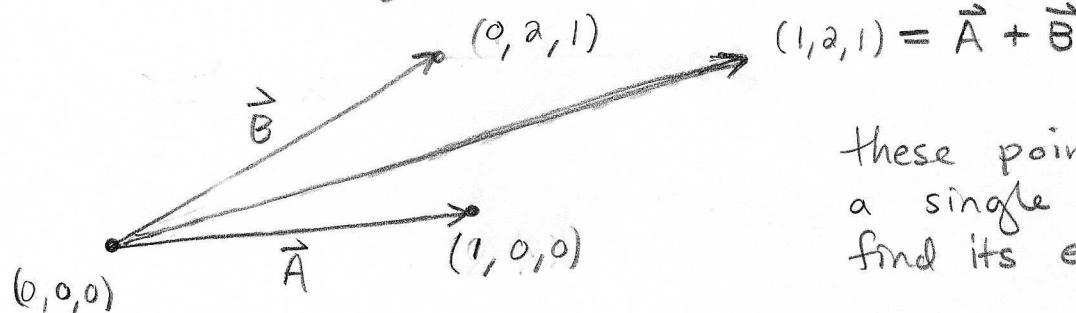
Thus, using Stoke's Th^m,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot (r dr d\theta \hat{k}), \text{ where } z = 5, 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi, \\ &= \int_0^{2\pi} \int_0^4 5r dr d\theta, \quad \hat{k} \text{ picks off the } z\text{-component of } \nabla \times \vec{F} \text{ which is nice since I'd rather not integrate the other two components.} \\ &= (2\pi) \frac{5r^2}{2} \Big|_0^4 \\ &= \boxed{80\pi}\end{aligned}$$

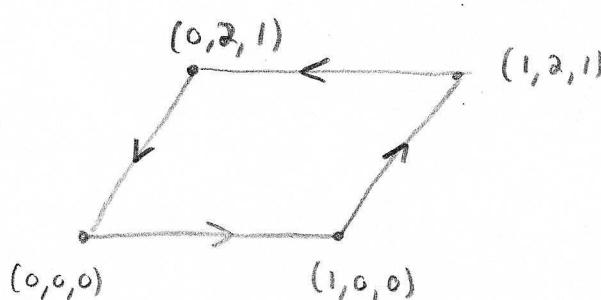
§17.8 #17] A particle moves along line segments from the origin (3)
to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$ and back to $(0, 0, 0)$
under the influence of a force field

$$\vec{F}(x, y, z) = \langle z^2, 2xy, 4y^2 \rangle$$

Find the work done by \vec{F} .



these points lie on
a single plane. Let's
find its eqⁿ.



$$\begin{aligned}\vec{A} &= \langle 1, 0, 0 \rangle \\ \vec{B} &= \langle 0, 2, 1 \rangle \\ \vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \langle 0, -1, 2 \rangle\end{aligned}$$

$$\Sigma(x, y) = \langle x, y, \frac{1}{2}y \rangle$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 2$

$$\text{has } dS = (\Sigma_x \times \Sigma_y) dx dy = \langle 0, -1/2, 1 \rangle dx dy$$

$$\Rightarrow -y + 2z = 0$$

$$\text{that is } z = \frac{1}{2}y$$

Now since I'm determined to use Stoke's Th^m to calculate
 $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ let's calculate,

$$\begin{aligned}\nabla \times \vec{F} &= \left\langle \frac{\partial}{\partial y}(4y^2) - \frac{\partial}{\partial z}(2xy), \frac{\partial}{\partial z}(z^2) - \frac{\partial}{\partial x}(4y^2), \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(z^2) \right\rangle \\ &= \langle 8y, 2z, 2y \rangle\end{aligned}$$

Hence, noting $2z = 2(\frac{1}{2}y) = y$ on S ,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \int_0^2 \langle 8y, y, 2y \rangle \cdot \langle 0, -1/2, 1 \rangle dy dx \\ &= \int_0^1 \int_0^2 (-y/2 + 2y) dy dx = \int_0^2 \frac{3y^2}{2} dy = \frac{3y^3}{6} \Big|_0^2 = 3\end{aligned}$$

Remark: Maybe direct computation of $\int_C \vec{F} \cdot d\vec{r}$ is the same trouble.

(4)

§ 17.9 #5 Use divergence Th^m to calculate $\iint_S \vec{F} \cdot d\vec{s}$ for S the box bounded by $x=0, x=1, y=0, y=1, z=0, z=2$ and $\vec{F}(x, y, z) = \langle e^x \sin(y), e^x \cos(y), yz^2 \rangle$

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(e^x \sin(y)) + \frac{\partial}{\partial y}(e^x \cos(y)) + \frac{\partial}{\partial z}(yz^2) \\ &= e^x \sin(y) - e^x \sin(y) + 2yz\end{aligned}$$

Thus, by the divergence Th^m we find,

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{s} &= \int_0^1 \int_0^1 \int_0^2 2yz \, dz \, dy \, dx \\ &= \int_0^1 yz^2 \Big|_0^2 \, dy = \\ &= \int_0^1 4y \, dy \\ &= 2y^2 \Big|_0^1 \\ &= \boxed{2}\end{aligned}$$

§ 17.9 #9 Let $\vec{F}(x, y, z) = \langle x \sin(z), \cos(xz), y \cos(z) \rangle$ find $\iint_S \vec{F} \cdot d\vec{s}$ where S is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Let E be the solid region with $S = \partial E$ (boundary of E is the ellipsoid S)

$$\nabla \cdot \vec{F} = y \sin(z) - y \sin(z) = 0$$

Well isn't that convenient,

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_E (\nabla \cdot \vec{F}) \, dV = \iiint_E 0 \, dV = \boxed{0}$$