

Antiderivatives (§4.9)

(elaborating ⑩ again, with derivatives to compare.) 91

Defn/ F is the antiderivative of f if $F'(x) = f(x)$. The most general antiderivative is the indefinite integral $\int f(x) dx$ meaning

$$\frac{d}{dx} \left(\int f(x) dx \right) = f(x)$$

I list below all the basic antiderivatives and they're corresponding derivatives

$\int dx = x + C$	$\frac{d}{dx}(x + C) = \frac{d}{dx}(x) + \frac{d}{dx}(C) = 1$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\frac{d}{dx}(x^n) = nx^{n-1}$
$\int \frac{1}{x} dx = \ln x + C$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
$\int \cos(x) dx = \sin(x) + C$	$\frac{d}{dx}(\sin(x)) = \cos(x)$
$\int \sin(x) dx = -\cos(x) + C$	$\frac{d}{dx}(-\cos(x)) = -\sin(x)$
$\int \sec^2(x) dx = \tan(x) + C$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$
$\int \sec(x) \tan(x) dx = \sec(x) + C$	$\frac{d}{dx}(\sec(x)) = \sec(x) \tan(x)$
$\int \csc^2(x) dx = -\cot(x) + C$	$\frac{d}{dx}(-\cot(x)) = -\csc^2(x)$
$\int \csc(x) \cot(x) dx = -\csc(x) + C$	$\frac{d}{dx}(-\csc(x)) = -\csc(x) \cot(x)$
$\int e^x dx = e^x + C$	$\frac{d}{dx}(e^x) = e^x$
$\int a^x dx = \frac{1}{\ln(a)} a^x + C$	$\frac{d}{dx}(a^x) = \ln(a) a^x$
$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$	$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$
$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$	$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

You should memorize every integral on this page. Additionally know that:

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$\int c f(x) dx = c \int f(x) dx$$

INTEGRATION BY U-Substitution (55.5)

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This is the simplest technique of integration, although its 1st step, namely picking u , is not always obvious. Practice makes perfect, but even before that review your basic integrals!

E1 $\int x e^{x^2} dx = \int x e^u \frac{du}{2x} \leftarrow \begin{array}{|l} u = x^2 \\ \frac{du}{dx} = 2x \therefore dx = \frac{du}{2x} \end{array}$

$$\begin{aligned} &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \boxed{\frac{1}{2} e^{x^2} + C} \end{aligned}$$

E2 $\int (ax+b)^{13} dx = \int u^{13} \frac{du}{a} \leftarrow \begin{array}{|l} u = ax+b \\ \frac{du}{dx} = a \therefore dx = \frac{du}{a} \end{array}$

$$\begin{aligned} &= \frac{1}{14a} u^{14} + C \\ &= \boxed{\frac{1}{14a} (ax+b)^{14} + C} \end{aligned}$$

E3 $\int 5^{\frac{x}{3}} dx = \int 5^u (3du) \leftarrow \begin{array}{|l} u = \frac{1}{3}x \\ \frac{du}{dx} = \frac{1}{3} \therefore dx = 3du \end{array}$

$$\begin{aligned} &= 3 \frac{5^u}{\ln(5)} + C \\ &= \boxed{\frac{3}{\ln(5)} 5^{\frac{x}{3}} + C} \end{aligned}$$

E4

$$\begin{aligned}
 \int \tan(x) dx &= \int \frac{\sin(x)}{\cos(x)} dx \\
 &= - \int \frac{\sin(x)}{u} \left(-\frac{du}{\sin(x)} \right) \\
 &= - \int \frac{1}{u} du \\
 &= -\ln|u| + C \\
 &= -\ln|\cos(x)| + C = \ln|\sec(x)| + C
 \end{aligned}$$

$$u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x) \therefore dx = -\frac{du}{\sin(x)}$$

E5

$$\begin{aligned}
 \int \frac{2x}{1+x^4} dx &= \int \frac{2x}{1+u^2} \frac{du}{2x} \\
 &= \int \frac{du}{1+u^2} \\
 &= \tan^{-1}(u) + C \\
 &= \boxed{\tan^{-1}(x^2) + C}
 \end{aligned}$$

$$u = x^2$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

You can also say "you can calculate this integral using sub."

$$\begin{aligned}
 &\int \frac{dx}{1+x^2} \quad \text{Let } x = \tan u \\
 &= \int 1 du \quad dx = \sec^2 u du \\
 &= u = \tan^{-1} x
 \end{aligned}$$

(looking ahead)

E6

$$\begin{aligned}
 \int \sqrt[3]{1-3x} dx &= \int \sqrt[3]{u} \frac{du}{-3} \\
 &= -\frac{1}{3} \frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C \\
 &= -\frac{1}{3} \frac{3}{4} u^{\frac{4}{3}} + C \\
 &= \boxed{-\frac{1}{4} (1-3x)^{\frac{4}{3}} + C}
 \end{aligned}$$

$$u = 1-3x$$

$$\frac{du}{dx} = -3 \therefore dx = \frac{du}{-3}$$

E7

$$\begin{aligned}
 \int \frac{1}{x+b} dx &= \int \frac{du}{u} \\
 &= \ln|u| + C \\
 &= \boxed{\ln|x+b| + C}
 \end{aligned}$$

$$u = x+b$$

$$\frac{du}{dx} = 1 \therefore dx = du$$

E8] $\int \frac{x^2}{\sqrt{x^2-x^4}} dx = \int \frac{x^2}{x\sqrt{1-x^2}} dx$

 $= \int \frac{x}{\sqrt{1-x^2}} dx$
 $= \int \frac{x}{\sqrt{u}} \frac{du}{-2x} \quad \leftarrow \begin{array}{|l} u=1-x^2 \\ \frac{du}{dx}=-2x \therefore dx=\frac{du}{-2x} \end{array}$
 $= -\frac{1}{2} \int u^{-\frac{1}{2}} du$
 $= -\frac{1}{2} \cdot 2u^{\frac{1}{2}} + C$
 $= \boxed{-\sqrt{1-x^2} + C}$

E9] $\int \frac{\ln(x)}{x} dx = \int \frac{u}{x} x du \quad \leftarrow \begin{array}{|l} u=\ln(x) \\ \frac{du}{dx}=\frac{1}{x} \therefore dx=x du \end{array}$

 $= \int u du$
 $= \frac{1}{2}u^2 + C$
 $= \boxed{\frac{1}{2}(\ln(x))^2 + C}$

E10] $\int \sin(3\theta) d\theta = \int \sin(u) \frac{du}{3} \quad \leftarrow \begin{array}{|l} u=3\theta \\ \frac{du}{d\theta}=3 \therefore d\theta=\frac{du}{3} \end{array}$

 $= -\frac{1}{3} \cos(u) + C$
 $= \boxed{-\frac{1}{3} \cos(3\theta) + C}$

E11] $\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx = \int \frac{u}{\sqrt{1-x^2}} (\sqrt{1-x^2} du) \quad \begin{array}{l} u=\sin^{-1}(x) \\ \frac{du}{dx}=\frac{1}{\sqrt{1-x^2}} \therefore dx=\sqrt{1-x^2} du \end{array}$

 $= \int u du$
 $= \frac{1}{2}u^2 + C$
 $= \boxed{\frac{1}{2}(\sin^{-1}(x))^2 + C}$

Remark:
 $x = \sin(u)$
 will work.

E12

$$\int t \cos(t^2 + \pi) dt = \int t \cos(u) \frac{du}{2t} \quad \leftarrow \begin{cases} u = t^2 + \pi \\ \frac{du}{dt} = 2t \therefore dt = \frac{du}{2t} \end{cases}$$

$$= \frac{1}{2} \int \cos(u) du$$

$$= \frac{1}{2} \sin(u) + C$$

$$= \boxed{\frac{1}{2} \sin(t^2 + \pi) + C}$$

E13

$$\int \sin^3 \theta d\theta = \int (1 - \cos^2 \theta) \sin \theta d\theta$$

$$= \int (1 - u^2) \sin \theta \frac{du}{-\sin \theta} \quad \leftarrow \begin{cases} u = \cos \theta \\ \frac{du}{d\theta} = -\sin \theta \therefore d\theta = \frac{du}{-\sin \theta} \end{cases}$$

$$= \int (u^2 - 1) du$$

$$= \frac{1}{3} u^3 - u + C$$

$$= \boxed{\frac{1}{3} \cos^3 \theta - \cos \theta + C}$$

E14

$$\int \frac{1}{a^2 + x^2} dx = \int \frac{1/a^2}{a^2/a^2 + x^2/a^2} dx, \text{ assume } a \neq 0$$

$$= \frac{1}{a^2} \int \frac{1}{1 + (\frac{x}{a})^2} dx$$

$$= \frac{1}{a^2} \int \frac{1}{1+u^2} adu$$

$$= \frac{1}{a} \tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$\leftarrow \begin{cases} u = x/a \\ \frac{du}{dx} = \frac{1}{a} \Rightarrow dx = adu \end{cases}$$

Definite Integrals involving U-substitution

There are two ways to do these

- i.) Find the antiderivative then use FTC to evaluate in x (bounds in x)
- ii.) Do the u-subst. and evaluate in u-variable (bounds in u)

Let's see how this goes, I'll illustrate ii) first recall E12

E15

$$\begin{aligned} \int_0^{\sqrt{\pi/2}} t \cos(t^2 + \pi) dt &= \left[\frac{1}{2} \sin(t^2 + \pi) + C \right]_0^{\sqrt{\pi/2}} \\ &= \left[\frac{1}{2} \sin\left(\frac{\pi}{2} + \pi\right) + C \right] - \left[\frac{1}{2} \sin(\pi) + C \right] \\ &= \frac{1}{2} \sin\left(\frac{3\pi}{2}\right) \\ &= \boxed{-\frac{1}{2}} \end{aligned}$$

Alternatively using method ii)

$$\begin{aligned} E16 \quad \int_0^{\sqrt{\pi/2}} t \cos(t^2 + \pi) dt &= \int_{\pi}^{3\pi/2} t \cos(u) \frac{du}{2t} \\ &= \frac{1}{2} \int_{\pi}^{3\pi/2} \cos(u) du \\ &= \frac{1}{2} \sin(u) \Big|_{\pi}^{3\pi/2} \\ &= \frac{1}{2} \sin\left(\frac{3\pi}{2}\right) - \frac{1}{2} \sin(\pi) \\ &= \boxed{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} u &= u(t) = t^2 + \pi \\ \frac{du}{dt} &= 2t \therefore dt = \frac{du}{2t} \\ u(0) &= \pi \\ u(\sqrt{\pi/2}) &= \frac{\pi}{2} + \pi = \frac{3\pi}{2} \end{aligned}$$

The bounds must change when we change the variable of integration.

E17

$$\begin{aligned} \int_{4\pi^2}^{9\pi^2} \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= \int_{2\pi}^{3\pi} \frac{\sin(u)}{\sqrt{x}} 2\sqrt{x} du \\ &= \int_{2\pi}^{3\pi} 2\sin(u) du \\ &= -2\cos(u) \Big|_{2\pi}^{3\pi} \\ &= -2\cos(3\pi) + 2\cos(2\pi) \\ &= \boxed{4} \end{aligned}$$

$$\begin{aligned} u &= u(x) = \sqrt{x} \\ \frac{du}{dx} &= \frac{1}{2\sqrt{x}} \therefore dx = 2\sqrt{x} du \\ u(4) &= \sqrt{4\pi^2} = 2\pi \\ u(9) &= \sqrt{9\pi^2} = 3\pi \end{aligned}$$

E18/

$$\int_0^{\pi/4} \tan^3 \theta d\theta = \left[\ln |\cos \theta| + \frac{1}{2 \cos^2 \theta} \right]_0^{\pi/4} = \left(\ln \left| \frac{\sqrt{2}}{2} \right| + 1 \right) - \left(\ln(1) + \frac{1}{2} \right) = \boxed{\ln \left(\frac{\sqrt{2}}{2} \right) + \frac{1}{2}}$$

Using work below.

$$\int \tan^3 \theta d\theta = \int \frac{\sin^3 \theta}{\cos^3 \theta} d\theta$$

$$= \int \frac{1 - \cos^2 \theta}{\cos^3 \theta} \sin \theta d\theta$$

$$= \int \frac{1 - u^2}{u^3} (-du)$$

$$\begin{aligned} u &= \cos \theta \\ \frac{du}{d\theta} &= -\sin \theta \Rightarrow -du = \sin \theta d\theta \end{aligned}$$

$$= \int \left(\frac{1}{u} - \frac{1}{u^3} \right) du$$

$$= \ln |u| + \frac{1}{2u^2} + C$$

$$= \boxed{\ln |\cos \theta| + \frac{1}{2 \cos^2 \theta} + C}$$

$$E19/ \tan^2 \theta + 1 = \sec^2 \theta \quad \text{and} \quad \frac{d}{d\theta} (\tan \theta) = \sec^2 \theta$$

$$\int \sec^6 \theta d\theta = \int \sec^4 \theta \sec^2 \theta d\theta$$

$$= \int (1 + \tan^2 \theta)^2 \sec^2 \theta d\theta$$

$$= \int (1 + u^2)^2 du$$

$$\begin{aligned} u &= \tan \theta \\ \frac{du}{d\theta} &= \sec^2 \theta \Rightarrow \sec^2 \theta d\theta = du \end{aligned}$$

$$= \int (1 + 2u^2 + u^4) du$$

$$= u + \frac{2}{3} u^3 + \frac{1}{5} u^5 + C$$

$$= \boxed{\tan \theta + \frac{2}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta + C}$$

So we can then calculate definite integrals using the above result,

$$\int_0^{\pi/4} \sec^6 \theta d\theta = \left[\tan \theta + \frac{2}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta \right]_0^{\pi/4}$$

$$= \left(\tan(\pi/4) + \frac{2}{3} \tan^3(\pi/4) + \frac{1}{5} \tan^5(\pi/4) \right) - 0$$

since $\tan(0) = 0$
and $\tan(\pi/4) = 1$

$$= 1 + \frac{2}{3} + \frac{1}{5}$$

$$= \frac{15 + 10 + 3}{15}$$

$$= \boxed{\frac{28}{15}}$$

E20 The easiest solⁿ is often the most clever one,

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$$\int \sec(\theta) d\theta = \int \frac{du}{u} \quad \xleftarrow{\text{see below* for why}}$$

$u = \sec \theta + \tan \theta$
 $\frac{du}{u} = \sec \theta d\theta$

$$= \ln|u| + C$$
$$= \boxed{\ln|\sec \theta + \tan \theta| + C}$$

* Where the substitution followed from

$$\frac{d}{d\theta}(\sec \theta + \tan \theta) = \sec \theta \tan \theta + \sec^2 \theta = \sec \theta (\tan \theta + \sec \theta)$$

In other words since $u = \sec \theta + \tan \theta$,

$$\frac{du}{d\theta} = (\sec \theta)u \Rightarrow \frac{du}{u} = \sec \theta d\theta \quad (\text{as we claimed above}).$$

* There are other ways to do this integral, but this is by far the most efficient solⁿ: See below the less inspired method,

(Ignore till later after you've studied "partial fractions")

$$\begin{aligned} \int \sec \theta d\theta &= \int \frac{1}{\cos \theta} d\theta \\ &= \int \frac{\cos \theta d\theta}{\cos^2 \theta} \\ &= \int \frac{\cos \theta d\theta}{1 - \sin^2 \theta} \\ &= \int \frac{1}{1-u^2} du \quad \xrightarrow{\substack{u = \sin \theta \\ du = \cos \theta d\theta}} \\ &= \frac{1}{2} \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du \quad \leftarrow \boxed{\text{Partial Fraction Decomposition.}} \\ &= \frac{1}{2} \left(\ln|1+u| - \ln|1-u| \right) + C \\ &= \boxed{\frac{1}{2} \left(\ln|1+\sin \theta| - \ln|1-\sin \theta| \right) + C} \end{aligned}$$

Clearly the u-substitution is easier, if you know to pick it.

Bonus Point: Show that the two answers are the same despite their apparent difference

(Hand-in to me soon please.)