

Numerical Methods: Given $f(x)$ which is analytic we can approximate $f(x)$ near $x=a$ by

$$f(x) \approx T_n(x)$$

Where $T_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$. The natural questions that arise are.

- ① for a particular n how good an approx is $T_n(x)$ to $f(x)$?
- ② if we desire a certain accuracy for $f(x)$ what is the minimum n for which $f(x) \approx T_n(x)$

Both of these questions can be answered if we know $|R_n(x)| = |f(x) - T_n(x)|$. Which is possible to estimate

- ① graphically (a bit cheesy assumes we can graph $f(x)$ right?)
- ② Alt. Series Thm (much better, don't need the answer to get the answer!)
- ③ Taylor's Ineq $0 \leq |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$
when $|f^{(n+1)}(x)| \leq M$ for $|x-a| < R$.

E1 For what interval about zero can we approx $\sin(x)$ by x to two decimals?

$$\sin(x) = x - \frac{1}{3!}x^3 + \dots \leftarrow \text{alternating series.}$$

If we keep upto x then $\sin(x) \approx x$ upto an error $\leq \frac{|x|^3}{3!}$
the max error is

$$\text{error} = \frac{|x|^3}{3!} = 0.01 \Rightarrow |x| = \sqrt[3]{0.06} = 0.39$$

That means $\sin \theta \approx \theta$ for $-0.39 \leq \theta \leq 0.39$

In degrees $|\theta| \leq 22.3^\circ$.

E2 So we're faced with the task of accurately calculating the $\sqrt{4.03}$ to seven decimals. For the purposes of this example assume all calculators are evil, it's after the robot holocaust so they can't be trusted. What to do? We'll use E8 on 225 plus the Alternating Series estimation theorem,

$$\sqrt{x} = \underbrace{2 + \frac{1}{4}(x-4)}_{\text{nothing new}} - \underbrace{\frac{1}{64}(x-4)^2}_{\text{thank you Mr. Taylor}} + \underbrace{\frac{1}{512}(x-4)^3}_{\text{this makes it even better}} + \dots$$

see pg. 54

than a linear approximation.

Comments aside lets calculate, try using 1st 3 terms,

$$\begin{aligned}\sqrt{4.03} &= 2 + \frac{1}{4}(4.03-4) - \frac{1}{64}(4.03-4)^2 \\ &= 2 + \frac{1}{4}(0.03) - \frac{1}{64}(0.03)^2 \\ &= 2 + \frac{3}{4} \cdot \frac{1}{100} - \frac{9}{64} \left(\frac{1}{100}\right)^2 \\ &= 2.00 + 0.0075 - 0.000014062 \\ &= 2.007485938\end{aligned}$$

How many of these digits are certain?
Well the series is alternating thus we know the error is smaller than the next term in the series,

$$\text{Error} \leq \frac{1}{512}(0.03)^3 = \frac{27}{512} \left(\frac{1}{100}\right)^3 \approx 0.000000058$$

Thus $\sqrt{4.03} = 2.007485938 \pm 0.000000058$

For sure $\boxed{\sqrt{4.03} \approx 2.0074859}$ the next digit 3 is uncertain.

Compare this to $\sqrt{4.03} = 2.00748598999$ from my TI-89

not to shabby, though I'm not sure.

arithmetic.

$$\begin{array}{r} 0.14062 \\ 64 \overline{) 9.0} \\ 64 \\ \hline 260 \\ 256 \\ \hline 400 \\ 384 \\ \hline 160 \\ 0.007500000 \\ 0.000014062 \\ \hline 0.007485938 \end{array}$$

$$\begin{array}{r} 0.058 \\ 512 \overline{) 27.00} \\ 25.60 \\ \hline 4400 \\ 4112 \end{array}$$

Using Series to Integrate

Back in chapter 5 we pretty much covered the ways you can solve integrals in terms of elementary functions.

There are cases for which no simple antiderivative can be found for example you cannot "solve" the following,

$$\int \frac{e^x}{x} dx \quad \int \sin(x^2) dx \quad \int \cos(e^x) dx$$

$$\int \sqrt{x^3 + 1} dx \quad \int \frac{1}{\ln(x)} dx \quad \int \frac{\sin(x)}{x} dx$$

Well we cannot "solve" these as we did in chapter 5. However we can find power series solutions to these. Sometimes we'll only need to find the first few nontrivial terms, that alone can be quite useful to an application where an integral like one of the above arises. (E7 on 218 is similarly motivated)

E1 Find the complete power series sol^{1/2} to $\int \frac{e^x}{x} dx$.

$$\frac{e^x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{x^{n-1}}{n!} \right)$$

Now we can integrate term by term, (not at zero!)

$$\begin{aligned} \int \frac{e^x}{x} dx &= \int \left(\sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} \right) dx \\ &= \sum_{n=0}^{\infty} \int \frac{x^{n-1}}{n!} dx + C \\ &= \int \frac{1}{x} dx + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!} dx + C \\ &= \boxed{\ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n n!} + C} \end{aligned}$$

Lets differentiate and see if it works, is this really the antiderivative?

$$\begin{aligned} \frac{d}{dx} \left(\ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n n!} \right) &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{x^n}{n n!} \right) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \\ &= \frac{1}{x} \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \right) = \frac{e^x}{x} \end{aligned}$$

Using series to solve integrals: Power Series Sol^{1/2}s to integrals

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E2] Find power series sol^{1/2} to $\int \sin(x^2) dx$.

$$\begin{aligned}\sin(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} : \text{using the MacLaurin series for sine which we know.} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sin(x^2)\end{aligned}$$

Now use the series to represent the integrand (just as we did in E1)

$$\begin{aligned}\int \sin(x^2) dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} dx \\ &= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{4n+2}}{(2n+1)!} dx + C \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{4n+2} dx + C \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3} x^{4n+3} + C} \leftarrow \text{complete sol}^1 \\ &= \boxed{C + \frac{1}{3}x^3 - \frac{1}{3!} \frac{1}{7}x^7 + \frac{1}{5!} \frac{1}{11}x^{11} + \dots} \leftarrow \begin{matrix} \text{1st 3} \\ \text{non-trivial terms} \end{matrix}\end{aligned}$$

E3] $\int \cos(e^x) dx$, hmm. this one is more interesting!

$$\cos(e^x) = \sum_{n=0}^{\infty} (-1)^n \frac{(e^x)^{2n}}{(2n)!} \quad \text{is this ok? Why or why not?}$$

$$\int \cos(e^x) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(e^x)^{2n}}{(2n)!} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \int e^{2nx} dx + C$$

$$= \int e^x dx + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} \int e^{2nx} dx + C$$

$$= \boxed{C + x + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} \frac{1}{2n} e^{2nx}} \leftarrow \text{complete sol}^1$$

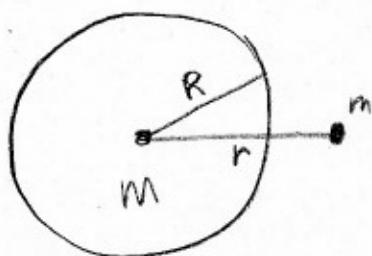
$$= \boxed{C + x - \frac{1}{4} e^{2x} + \frac{1}{4!} \frac{1}{4} e^{4x} + \dots} \leftarrow \begin{matrix} \text{1st 3} \\ \text{non-trivial terms.} \end{matrix}$$

What is wrong with this example?

Bonus Point if you can clearly tell me

Applications to Physics

E1 Why does $PV = mgy$ for gravity? Isn't it the case that $V = -\frac{GmM}{r}$



$$V(R) = -\frac{GmM}{R}$$

$$V'(R) = \frac{GmM}{R^2}$$

$$V''(R) = -\frac{2GmM}{R^3}$$

Well taylor expand $V(r)$ about $r = R$

$$V(r) \approx -\frac{GmM}{R} + \frac{GmM}{R^2}(r-R) - \frac{GmM}{R^3}(r-R)^2 + \dots$$

Think about it, $r-R$ is height above ground so noting $\frac{GM}{R^2} = g = 9.81$.

$$V(y) \approx -\frac{GmM}{R} + mgy \quad \text{upto error of } \frac{GmM}{R^3} y^2$$

That is $PE = mgy$ near surface of earth



find \vec{E} field at the point P , assuming $x \gg d$.

$$E = \frac{q_1}{x^2} - \frac{q_2}{(x+d)^2}$$

Notice $\frac{1}{(x+d)^2} = \frac{1}{x^2(1 + d/x)^2} = \frac{1}{x^2} \left(1 + \frac{d}{x}\right)^{-2} \leftarrow$ Binomial Series for $\frac{d}{x} < 1$ as is the case.

$$\therefore \frac{1}{(x+d)^2} \cong \frac{1}{x^2} \left(1 - 2\frac{d}{x} + \dots\right)$$

$$\text{Thus } E = \frac{q_1}{x^2} - q_2 \left[\frac{1}{x^2} \left(1 - 2\frac{d}{x} + \dots\right) \right] = \boxed{\frac{2q_1 d}{x^3} = E}$$

this gives the "far-field" approximation, it shows how the dominant contribution of E scales with distance.

Special Relativity: for a particle with mass m_0

$$E = \gamma m_0 c^2 \quad (\text{Total Energy of free particle})$$

$$K = \gamma m_0 c^2 - m_0 c^2 \quad (\text{Relativistic Kinetic Energy})$$

Where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ with v = velocity of particle
 c = speed of light

Let's calculate the power series expansion of $E(v)$ and see what it means physically, recall we found

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1 + \frac{1}{2} \left(\frac{v}{c}\right)^2 + \frac{3}{8} \left(\frac{v}{c}\right)^4$$

Then we see if $v \approx 0$ we have

$$\begin{aligned} E &= \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \dots\right) m_0 c^2 \\ &= m_0 c^2 + \frac{1}{2} m_0 v^2 + \dots \end{aligned}$$

↑ ↑
 Rest Energy Usual Kinetic
 of Particle Energy.

It's easy to see that $K \approx \frac{1}{2} m_0 v^2$ for $|v| \ll c$, which is good, the Relativistic Kinetic Energy should become the ordinary Newtonian Kin. Energy when $|v| \ll c$, that is for "non-relativistic speeds". See the text for a precise discussion of the error here, Ex 3. in §8.9.

More SERIES in Physics (Digression!)

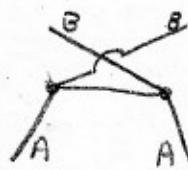
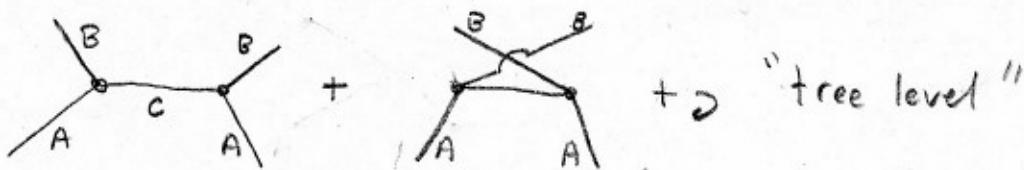
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TAYLOR SERIES are nice for known functions.

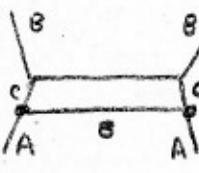
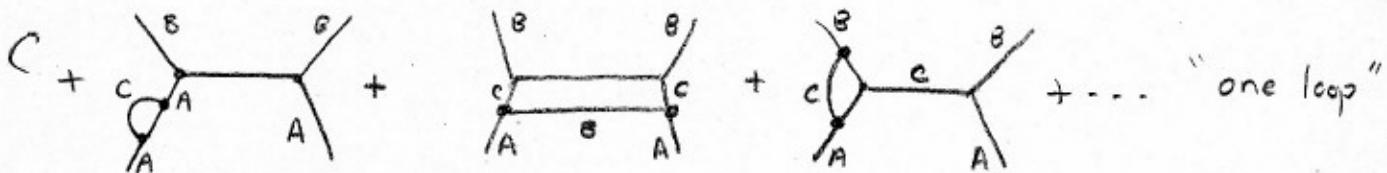
In modern physical theories the equations are so difficult to solve that we often have only a "perturbative" description of the physics.

What this means is we have to find a series to describe physical things (like how big a particle is, or its mass, ...). You might wonder how can we find the series, after all $f(x)$ is not known, well we use what are called "FEYNMAN DIAGRAMS" let me illustrate

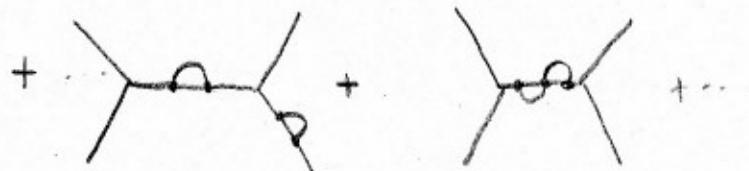
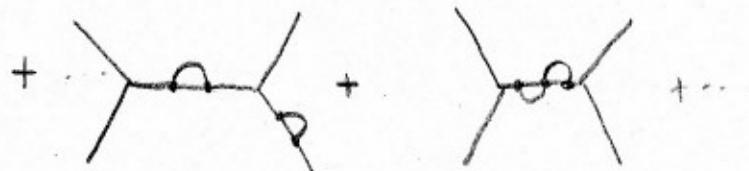
for $A + A \rightarrow B + B$ (two "A" particles collide and become two outgoing "B" particles)



"tree level"



"one loop"



+ ...

From such pictures one can make predictions about the physical characteristics of elementary particles.

The man who invented these diagrams had a van with them painted on long before anybody knew what they were, needless to say he lived in California.