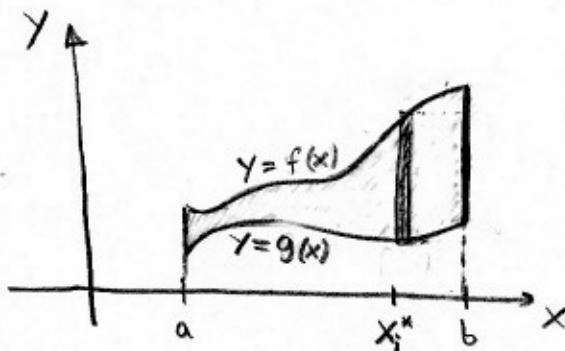


We have investigated how to find the "signed" area under the curve  $y = f(x)$ , we know that this is precisely what the definite integral gives us. Now we move onto a more general question, "what is the area bounded by some set of curves?"

**E1** Consider a generic example to begin, find the area bounded by  $x = a$ ,  $x = b$  and  $f(x)$  and  $g(x)$  where  $f(x) > g(x)$  for all  $x$  on  $[a, b]$ . A picture helps



we can find the area by dividing the area up into ~~only~~ many infinitesimal rectangles. We've drawn a typical box you can see it's height is  $f(x) - g(x)$ ; it has area  $\Delta A_i = [f(x_i) - g(x_i)]\Delta x$ .

$$A = \text{AREA} = \lim_{n \rightarrow \infty} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x$$

But this is precisely the integral of  $f(x) - g(x)$ ,

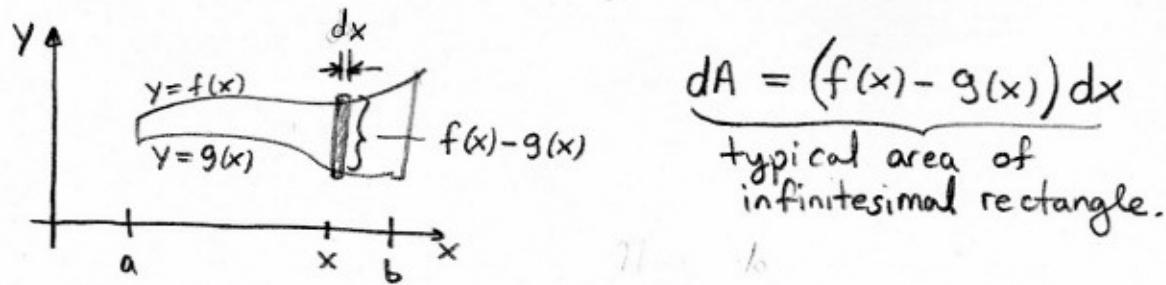
$$A = \int_a^b (f(x) - g(x)) dx$$

- Technically, we should begin all problems with arguments about  $\Delta x$  and  $\Delta A$ ; and so on, these are finite. Then once the object of interest is suitably approximated by  $n$ -rectangles we pass to the limit  $n \rightarrow \infty$  and find the  $\sum$  becomes an  $\int$  and  $\Delta x$  becomes a  $dx$ . For all calculational purposes the first steps with " $\Delta$ "'s are unnecessary. We can make these arguments using infinitesimals from the beginning and I will from now on. I mention this so you can understand the connection between the method in these notes and the more cumbersome arguments in your text

;) IMHO.

**E1** Revisited using infinitesimal arguments.

(133)



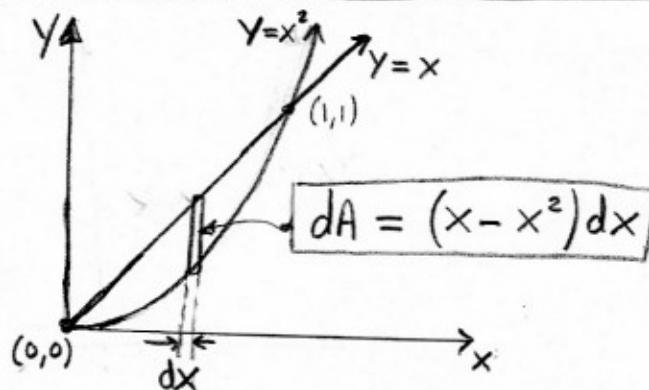
$$dA = (f(x) - g(x)) dx$$

typical area of  
infinitesimal rectangle.

To obtain area simply add up all the infinitesimal areas,

$$A = \int dA = \int_a^b (f(x) - g(x)) dx$$

**E2** Area bounded by  $y = x$  and  $y = x^2$  is what?



• Need to find intersection points  
here we can simply equate  $y$ ,

$$x = x^2$$

$$x^2 - x = x(x-1) = 0$$

$$x=0 \text{ or } x=1$$

The points of intersection  
are  $(0,0)$  and  $(1,1)$ .

$$\begin{aligned} A &= \int dA = \int_0^1 (x - x^2) dx \\ &= \left( \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \boxed{\frac{1}{6}} \end{aligned}$$

E3 find area bounded by  $y = x^2$  and  $y = \sqrt{x}$

Lets find where these curves intersect, set  $y = y$  yielding,

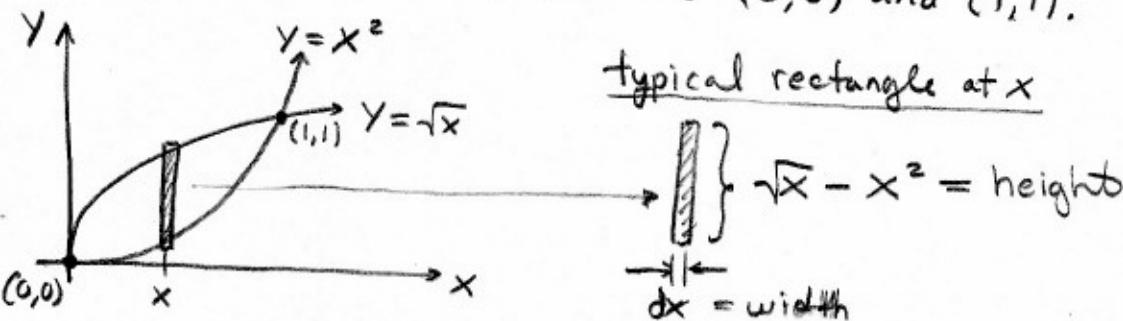
$$x^2 = \sqrt{x}$$

$$x^4 = x$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1$$

Thus the points of intersection are  $(0,0)$  and  $(1,1)$ .



The area of this tiny rectangle is,  $dA = (\sqrt{x} - x^2)dx$

$$\begin{aligned} \text{Thus } A &= \int_0^1 (\sqrt{x} - x^2)dx \\ &= \left( \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right) \Big|_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

### Strategy

- ① Graph region carefully, use algebra to find points of intersection.
- ② Draw a typical tiny rectangle and find its area( $dA$ ).
- ③ Add the areas of the tiny rectangles by integrating.

E4] Find area bounded by  $y^2 = x$  and  $2y = x - 3$

Notice that at intersection  $x = x \Rightarrow y^2 = 2y + 3$

$$\Rightarrow y^2 - 2y - 3 = 0$$

$$\Rightarrow (y-3)(y+1) = 0$$

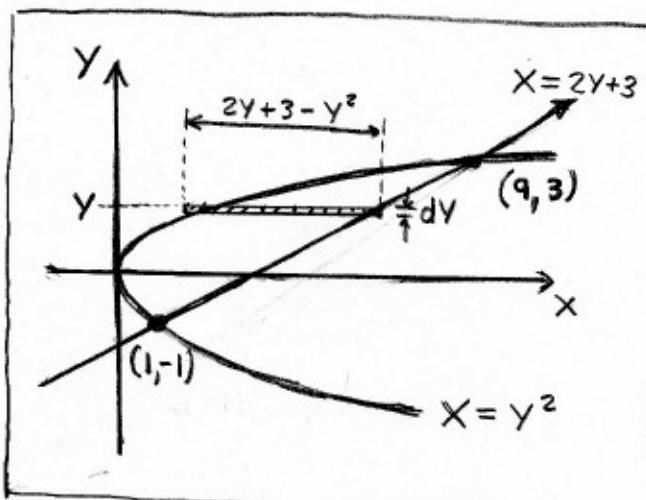
$$\Rightarrow \underline{y=3} \text{ or } \underline{y=-1}$$

$\Rightarrow$  points of intersection are  $(9, 3)$  and  $(1, -1)$

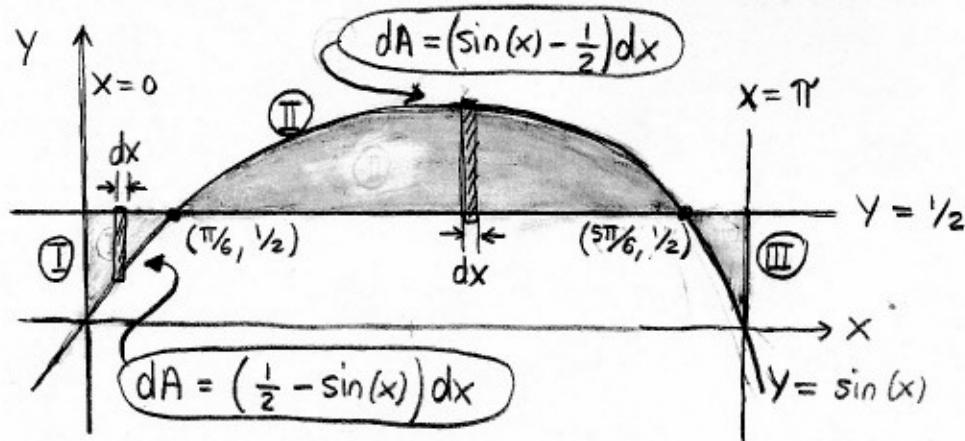
\* its better to use horizontal strips because we don't need to break up into cases. If we used vertical strips it would be tricky because  $0 \leq x \leq 1$  is different than  $1 \leq x \leq 9$  for vertical strips.

$$dA = (2y + 3 - y^2) dy$$

$$\begin{aligned} \text{Thus } A &= \int_{-1}^3 (2y + 3 - y^2) dy \\ &= \left( y^2 + 3y - \frac{1}{3}y^3 \right) \Big|_{-1}^3 \\ &= (9 + 9 - \frac{1}{3}(27)) - (1 - 3 + \frac{1}{3}) \\ &= 9 + \frac{5}{3} = \boxed{\frac{32}{3} \approx 10.67} \end{aligned}$$



[E5] Find area bounded by  $y = \sin(x)$  and  $y = \frac{1}{2}$  and  $x = 0$  and  $x = \pi$



Notice that  $\sin(30^\circ) = \sin(\pi/6) = \frac{1}{2}$  and by the symmetry of  $\sin(x)$  about  $x = \pi/2$  it's clear  $\sin(5\pi/6) = \frac{1}{2}$ . This reveals the intersection points are  $(\pi/6, \frac{1}{2})$  and  $(5\pi/6, \frac{1}{2})$ . Clearly we need to divide up into cases ①, ② and ③.

$$\begin{aligned}
 A &= \int_0^{\pi/6} \left(\frac{1}{2} - \sin(x)\right) dx + \int_{\pi/6}^{5\pi/6} \left(\sin(x) - \frac{1}{2}\right) dx + \int_{5\pi/6}^{\pi} \left(\frac{1}{2} - \sin(x)\right) dx \\
 &= \frac{\pi}{12} + \cos(x) \Big|_0^{\pi/6} - \frac{4\pi}{12} - \cos(x) \Big|_{\pi/6}^{5\pi/6} + \frac{\pi}{12} + \cos(x) \Big|_{5\pi/6}^{\pi} \\
 &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 - \frac{4\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2} \\
 &= -\frac{2\pi}{12} + 2\sqrt{3} - 2 \\
 &= 2(-\sqrt{3} - 1) - \pi/6 \\
 &\approx 0.9405 = A
 \end{aligned}$$

- Reasonable # considering  $\int_0^{\pi} \sin(x) dx = -\cos(x) \Big|_0^{\pi} = 2$   
It would seem my arithmetic is sound.



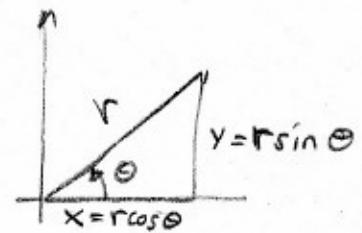
## Polar Coordinates : $(r, \theta)$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\theta = \tan^{-1}(\frac{y}{x})$$

$$r = \sqrt{x^2 + y^2}$$



Consider then some examples of Cartesian vs. Polar

$$x^2 + y^2 = a^2 \longleftrightarrow r = a$$

$$y = x \longleftrightarrow \theta = \pi/4$$

$$y = x + 1 \longleftrightarrow r \sin \theta = r \cos \theta + 1 \Rightarrow r(\sin \theta - \cos \theta) = 1$$

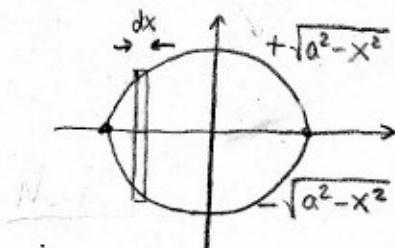
$$r = \frac{1}{\sin \theta - \cos \theta}$$

[E6] Lets find the area of a circle (again)

The eq's of the circle in parametric form are,

$$x = a \cos \theta \quad \text{and} \quad y = a \sin \theta$$

$$\text{Where } x^2 + y^2 = a^2 \Rightarrow y = \pm \sqrt{a^2 - x^2}$$



$$dA = 2\sqrt{a^2 - x^2} dx$$

$$A = \int_{-a}^a 2\sqrt{a^2 - x^2} dx$$

$$= \int_{-\pi}^0 2\sqrt{a^2 - a^2 \cos^2 \theta} (-a \sin \theta d\theta)$$

$$= 2a^2 \int_0^\pi \sin^2 \theta d\theta$$

$$= 2a^2 \left( \int_0^\pi \frac{1}{2}(1 - \cos 2\theta) d\theta \right)$$

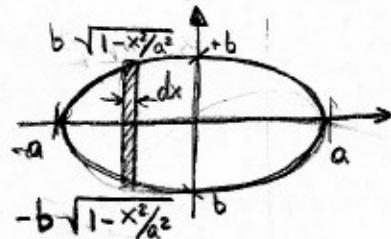
$$= 2a^2 \left( \frac{\pi}{2} - \cancel{\frac{\sin 2\theta}{4}} \Big|_0^\pi \right)$$

$$= \boxed{\pi a^2}$$

$$\begin{aligned} x &= a \cos \theta \\ dx &= -a \sin \theta d\theta \\ x = a &\Rightarrow \theta = 0 \\ x = -a &\Rightarrow \theta = \pi \end{aligned}$$

See pg. 445 Example 6 for another example. //

### E7 Area of Ellipse



Standard Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \pm b\sqrt{1-(x/a)^2}$$

Ellipse in Polar Coord.

$$x = a \cos \theta$$

$$y = b \sin \theta$$

$$dA = 2b\sqrt{1-(x/a)^2} dx$$

$$A = \int_{-a}^a 2b\sqrt{1-(x/a)^2} dx$$

$$= 2b \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 \theta} a \cos \theta d\theta$$

$$= 2ab \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta$$

$$= 2ab \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) d\theta$$

$$= ab \left[ \theta + \frac{\sin 2\theta}{2} \right]_{-\pi/2}^{\pi/2}$$

$$= ab \left[ \left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right) \right]$$

$$= \boxed{\pi ab}$$

$x = a \sin \theta$	$x = a \Rightarrow \theta = \pi/2$
$dx = a \cos \theta d\theta$	$x = -a \Rightarrow \theta = -\pi/2$

Notice when  $a=b=r$  we get  $A = \pi r^2$  which is good.

Digression: What is the eq<sup>n</sup> of this ellipse in polar coordinates?

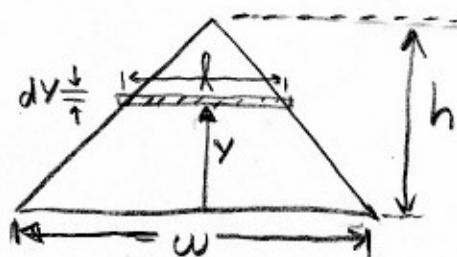
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} = 1$$

$$r^2 \left( \cos^2 \theta + \frac{a^2}{b^2} \sin^2 \theta \right) = a^2$$

$$\boxed{r = \frac{a}{\sqrt{\cos^2 \theta + \left(\frac{a}{b}\right)^2 \sin^2 \theta}}}$$

We can again notice when  $a=b$  we get  $r=a$  a very sensible eq<sup>n</sup> for a circle at the origin.

**E8** let's find the width of a slice in a triangular region of width  $w$  and height  $h$  as a function of  $y$ .



Again it's clear that for this shape that  $l$  varies linearly with  $y$ , so  $l = my + b$ . Additionally we know

$$y=0 \Rightarrow l=w=m(0)+b \therefore b=w$$

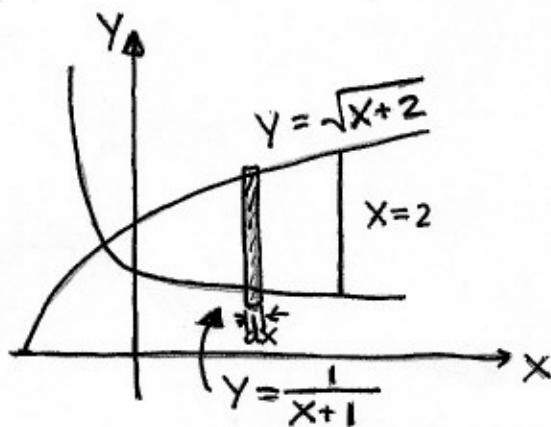
$$y=h \Rightarrow l=0=mh+w \therefore m=-w/h$$

Thus we find  $l = -\frac{w}{h}y + w$ .

We see that  $dA = l dy = \left(-\frac{w}{h}y + w\right) dy$  thus

$$\begin{aligned} A &= \int_0^h \left(-\frac{w}{h}y + w\right) dy \\ &= \left(-\frac{w}{2h}y^2 + wy\right) \Big|_0^h \\ &= -\frac{1}{2}wh + wh \\ &= \frac{1}{2}wh = \frac{1}{2}(\text{base})(\text{height}) \end{aligned}$$

E9

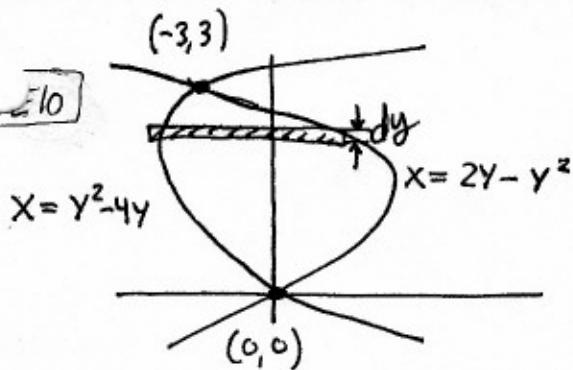


$$\begin{aligned} dA &= (y_t - y_b)dx \\ &= \left(\sqrt{x+2} - \frac{1}{x+1}\right)dx \end{aligned}$$

There is a vertical slice at each  $x$  from 0 to 2. Thus

$$\begin{aligned} A &= \int_0^2 \left( \sqrt{x+2} - \frac{1}{x+1} \right) dx = \left[ \frac{2}{3}(x+2)^{\frac{3}{2}} - \ln|x+1| \right]_0^2 \\ &= \left( \frac{2}{3}(4)^{\frac{3}{2}} - \ln(3) \right) - \left( \frac{2}{3}(2)^{\frac{3}{2}} - \ln(1) \right) \\ &= \boxed{2.349} \end{aligned}$$

E10



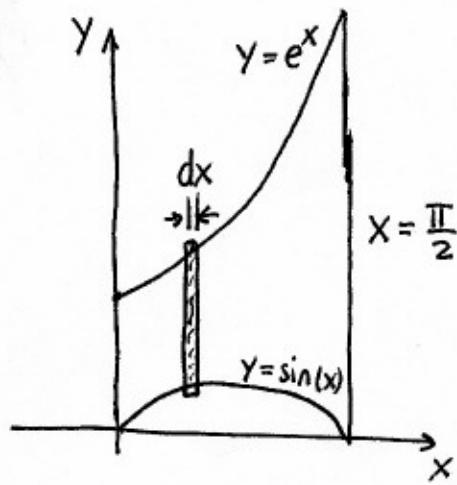
$$\begin{aligned} dA &= (x_r - x_l) dy \\ &= [(2y - y^2) - (y^2 - 4y)] dy \\ &= (6y - 2y^2) dy \end{aligned}$$

Notice it's easier to do it horizontally here, and also the integration simplifies if you simplify  $dA$  before integrating.

$$\begin{aligned} A &= \int_0^3 (6y - 2y^2) dy = \left[ 3y^2 - \frac{2}{3}y^3 \right]_0^3 \\ &= \left[ 3(9) - \frac{2}{3}(3)^3 \right] - [0 - 0] \\ &= (27 - \frac{2}{3}(27)) \\ &= \frac{1}{3}(27) \\ &= \boxed{9} \end{aligned}$$

E11

139c

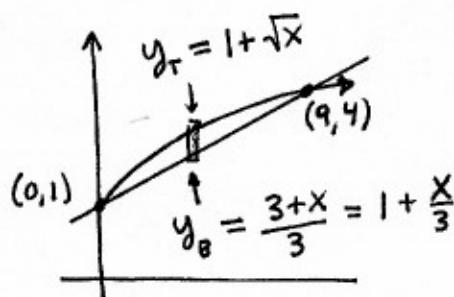


$$\begin{aligned} dA &= (y_T - y_B) dx \\ &= (e^x - \sin(x)) dx \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \int_0^{\pi/2} (e^x - \sin(x)) dx \\ &= [e^x + \cos(x)]_0^{\pi/2} = \boxed{e^{\pi/2} - 2 = 2.81} \end{aligned}$$

E12



$$\begin{aligned} dA &= (y_T - y_B) dx \\ &= (1 + \sqrt{x} - 1 - \frac{x}{3}) dx \\ &= (\sqrt{x} - \frac{x}{3}) dx \end{aligned}$$

Therefore,  
using algebra  
below to  
explain bounds,

$$\begin{aligned} A &= \int_0^9 (\sqrt{x} - \frac{x}{3}) dx \\ &= \left[ \frac{2}{3}x^{3/2} - \frac{1}{6}x^2 \right]_0^9 \\ &= \left[ \frac{2}{3}(9)^{3/2} - \frac{9^2}{6} \right] \\ &= \boxed{4.5} \end{aligned}$$

Points of intersection followed from  $y_B = y_T$  because,

$$1 + \sqrt{x} = 1 + \frac{x}{3}$$

$$\sqrt{x} = \frac{x}{3}$$

$$3\sqrt{x} = x$$

$$9x = x^2$$

$$x^2 - 9x = 0$$

$$x(x-9) = 0$$

$$\therefore \underline{x = 0 \text{ or } x = 9}$$

*Cannot just divide by x  
you lose information by doing  
that. (x might be zero.)*

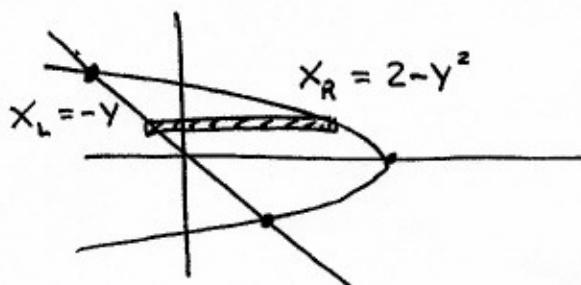
E13

$$x = 2 - y^2$$

$$x = -y$$

Just switch the roles of  $x$  &  $y$  to see why the graph should be a line and sideways parabola.

139d



Points of intersection follow from  $x_L = x_R$ ,

$$2 - y^2 = -y$$

$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0 \quad \therefore \underline{y=2} \text{ or } \underline{y=-1}$$

$$dA = (x_R - x_L) dy$$

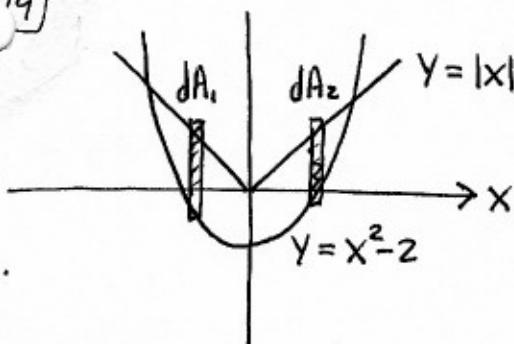
$$= (2 - y^2 - (-y)) dy$$

$$= (2 + y - y^2) dy$$

$$\therefore A = \int_{-1}^2 (2 + y - y^2) dy = \left[ 2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-1}^2$$

Evaluation yields  $\boxed{A = 4.5}$

r14



$$dA_1 = (-x - (x^2 - 2)) dx \quad (x < 0)$$

$$dA_2 = (x - (x^2 - 2)) dx \quad (x > 0)$$

### Points of intersection

$$\underline{x > 0} \quad x = x^2 - 2 \Rightarrow x^2 - x - 2 = (x-2)(x+1) = 0$$

$\therefore x = 2$  or  $(x \neq -1)$  throw that out because  $x \neq 0$ .

$$\underline{x < 0} \quad -x = x^2 - 2 \Rightarrow x^2 + x - 2 = (x+2)(x-1) = 0$$

$\therefore x = -2$  or  $(x \neq 1)$  throw out that sol<sup>n</sup> because  $x \neq 0$ .

$$A = \int_{-2}^0 (2 - x - x^2) dx + \int_0^2 (2 + x - x^2) dx$$

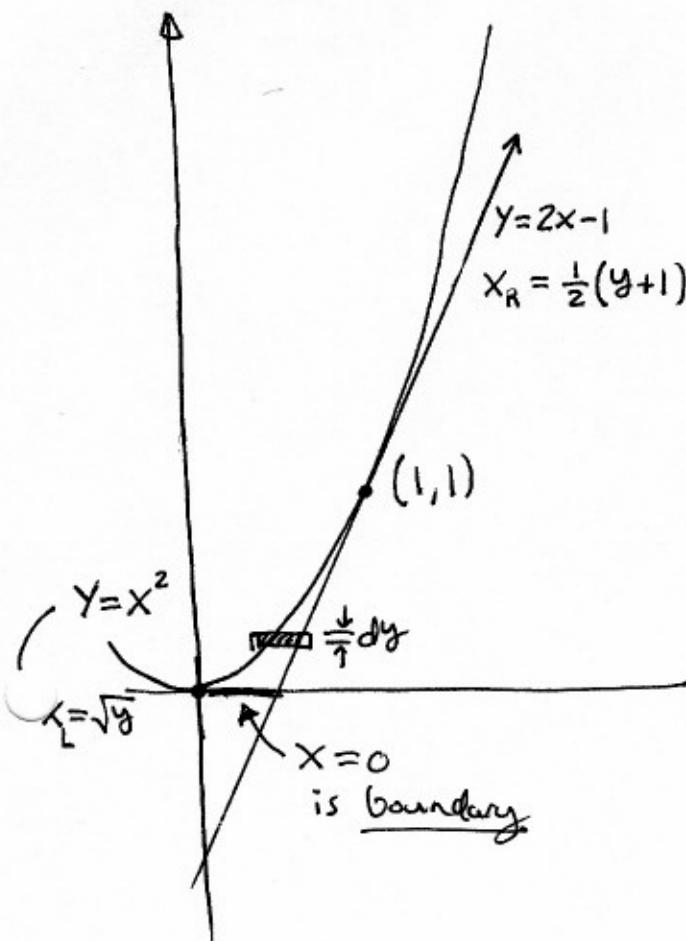
$$= \left( 2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_{-2}^0 + \left( 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^2 = \frac{10}{3} + \frac{10}{3} = \boxed{\frac{20}{3}}$$

E15 Find area bounded by  $y = x^2$  and the tangent to  $y = f(x) = x^2$  at  $(1, 1)$ , and the  $x$ -axis.  
 Recall we can write the tangent to  $y = f(x)$  at  $(a, f(a))$  in general (provided  $f'(a)$  exists)

$$y = f(a) + f'(a)(x - a)$$

Then we have  $a = 1$  and  $f'(x) = 2x \Rightarrow f'(1) = 2$  then

$$y = 1 + 2(x - 1) = \underline{2x - 1} = y \quad (\text{tangent line})$$



$$\begin{aligned} dA &= (x_R - x_L) dy \\ &= \left(\frac{1}{2}y + \frac{1}{2} - \sqrt{y}\right) dy \end{aligned}$$

Points of intersection:

$$2x - 1 = x^2$$

$$x^2 - 2x + 1 = (x - 1)^2 = 0$$

$\therefore \underline{x = 1} \Rightarrow y = 1$  at intersection  
 that is the pt. of intersection is  $(1, 1)$

Therefore, noting  $0 \leq y \leq 1$  for area,

$$\begin{aligned} A &= \int_0^1 \left( \frac{1}{2}y + \frac{1}{2} - \sqrt{y} \right) dy \\ &= \left( \frac{1}{4}y^2 + \frac{1}{2}y - \frac{2}{3}y^{3/2} \right) \Big|_0^1 = \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \boxed{\frac{1}{12}} \end{aligned}$$