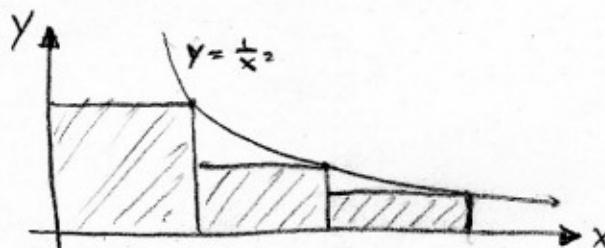


## More on the Question of Divergence §8.3 & §8.4

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I combine §8.3 & §8.4 and omit the error approx. until a later lecture! First, we'll complete our list of conv/div. tests. We already have two arguably three (geometric, telescoping and  $n^{\text{th}}$  term)

### Motivating Example 1:

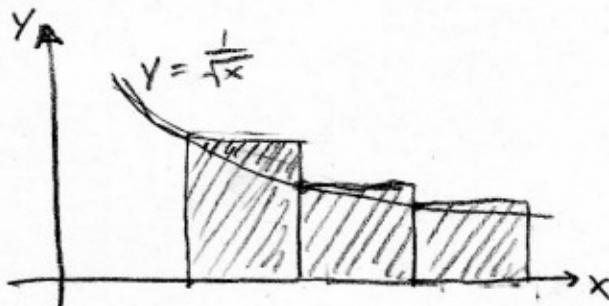


the boxed areas give  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

the curve  $y = \frac{1}{x^2}$  has the area under it given by  $\int_1^{\infty} \frac{1}{x^2} dx$   
(note we start at 1,  $\int_0^{\infty} \frac{1}{x^2} dx = \infty$ )

Remark: it seems reasonable to conclude that  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges since  $\int_1^{\infty} \frac{1}{x^2} dx = 1$  and is clearly bigger than the series, see graph above.

### Motivating Example 2:



the boxed areas give  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

the curve  $y = \frac{1}{\sqrt{x}}$  has area under it  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ .

Remark: if the  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$  diverges then surely the larger object  $\sum \frac{1}{\sqrt{n}}$  should also diverge.

### THM/(INTEGRAL TEST)

Suppose  $f$  is continuous, positive, decreasing fct on  $[1, \infty)$  with  $a_n = f(n)$ . Then the series  $\sum_{n=n_0}^{\infty} a_n$  is convergent  $\Leftrightarrow \int_{n_0}^{\infty} f(x) dx$  is convergent.

(a.) If  $\int_{n_0}^{\infty} f(x) dx$  converges  $\Rightarrow \sum_{n=n_0}^{\infty} a_n$  is convergent

(b.) If  $\int_{n_0}^{\infty} f(x) dx$  diverges  $\Rightarrow \sum_{n=n_0}^{\infty} a_n$  diverges.

Usually  $n_0 = 1$ .

Remark: if the series we're investigating corresponds to a function we can integrate then we can simply calculate an improper integral to determine the conv/div. of the series.

E11 Determine whether  $\sum_{n=1}^{\infty} ne^{-n^2}$  converges or diverges.

Notice that we can perform the integration  $\int xe^{-x^2} dx$ ,

$$\begin{aligned}\int xe^{-x^2} dx &= \int e^u \left(-\frac{du}{2}\right) && \leftarrow \begin{array}{|l} u = -x^2 \\ du = -2x dx \end{array} \\ &= \underbrace{-\frac{1}{2} e^{-x^2}}_{} + C\end{aligned}$$

Which makes it easy to compute the improper integral of interest,

$$\begin{aligned}\int_1^{\infty} xe^{-x^2} dx &= \lim_{t \rightarrow \infty} \left( \int_1^t xe^{-x^2} dx \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} e^{-x^2} \Big|_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} e^{-t^2} + \frac{1}{2} e^{-1} \right) \\ &= \boxed{\frac{1}{2e}} \quad \text{so it converges.}\end{aligned}$$

Therefore by integral test  $\sum_{n=1}^{\infty} ne^{-n^2}$  likewise converges.

E12

THE "P-SERIES TEST", show  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges } p > 1 \\ \text{diverges } p \leq 1 \end{cases}$

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We can use the integral test. First, assume  $p \neq 1$ ,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \left( \int_1^t x^{-p} dx \right) \\ &= \lim_{t \rightarrow \infty} \left( \frac{x^{-p+1}}{-p+1} \Big|_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) \\ &= \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p < 1 \end{cases} \quad \begin{matrix} (\text{Converges}) \\ (\text{Diverges}) \end{matrix} \end{aligned}$$

When  $p = 1$  something special happens,

$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t \rightarrow \infty} (\ln|x| \Big|_1^t) \\ &= \lim_{t \rightarrow \infty} (\ln(t)) \\ &= \infty \end{aligned}$$

Therefore by integral test  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges } p > 1 \\ \text{diverges } p \leq 1 \end{cases}$

Remark: The P-SERIES provides a benchmark for comparison. Given many examples we could compare to the P-series to see if conv/div. In this course we'll abstain from such argument. Any hwk. problem ought to be solvable by the tests I cover. You shouldn't need the "direct" or "limit comparison test". I have limited the scope of our analysis compared to some other sections of Ma 241. We also omitted the absolute convergence Thm for series.

### Th<sup>3</sup>/ ALTERNATING SERIES TEST (A.S.T.):

If you're given an alternating series,

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots, \quad b_n > 0$$

which satisfies the following two criteria

i.)  $b_{n+1} \leq b_n$  (for  $n \geq n_0 \geq 1$ )

ii)  $\lim_{n \rightarrow \infty} (b_n) = 0$

Then the series converges.

Remark: I've heard some call this the wishful thinking test. If your series alternates and decreases so that terms get small then the series converges. Perhaps nonintuitively this is not the case when the series is not alternating.

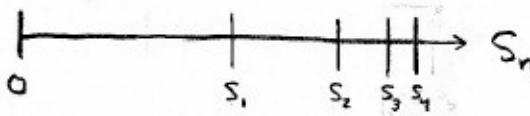
E/3  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is the "alternating harmonic series"

Notice  $b_n = \frac{1}{n} > 0$  and  $b_{n+1} \leq b_n$  since  $\frac{1}{n+1} \leq \frac{1}{n}$ .

Finally  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty \therefore$  By A.S.T.,  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  converged.

This is remarkable since the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges!

- An interesting picture can help us understand why this happens. I graph the partial sums of the  $\sum_{n=1}^{\infty} \frac{1}{n}$  and  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  to compare,



Harmonic Series keeps growing

$$S_1 = 1$$

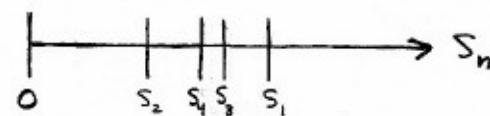
$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

⋮

$$S_{178,482,301} \approx 20$$



Alternating Harmonic series converges

$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

⋮

$$\lim_{n \rightarrow \infty} (S_n) = \ln(2) = 0.693.$$

E14  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$  is an alternating series with  $b_n = \frac{n^2}{n^3+1} > 0$

Notice (Extending  $n$  continuously) we can show  $b_n$  decreases,

$$\begin{aligned} \frac{d}{dn} \left( \frac{n^2}{n^3+1} \right) &= \frac{2n(n^3+1) - n^2(3n^2)}{(n^3+1)^3} \\ &= \frac{-n^4 + 2n}{(n^3+1)^3} \\ &= \frac{n}{(n^3+1)^3} (2 - n^3) < 0 \quad (\text{provided } n \geq 2) \end{aligned}$$

One criteria down, one to go,

$$\lim_{n \rightarrow \infty} \left( \frac{n^2}{n^3+1} \right) = \lim_{n \rightarrow \infty} \left( \frac{1/n}{1 + 1/n^3} \right) = 0$$

Thus by A.S.T the series converges.

E15

Does  $\sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1+2n}{n} \right)$  converge or diverge?

Clearly this is an alternating series,  $b_n = \frac{1+2n}{n}$  note then

$$\frac{d}{dn} \left( \frac{1+2n}{n} \right) = \frac{d}{dn} \left( \frac{1}{n} + 2 \right) = \frac{-1}{n^2} < 0 \Rightarrow b_n \text{ are decreasing}$$

But something is wrong, note  $\lim_{n \rightarrow \infty} \left( \frac{1+2n}{n} \right) \neq \lim_{n \rightarrow \infty} \left( \frac{2}{1} \right) = 2$ .

Thus the A.S.T. is inconclusive. Notice the  $n^{\text{th}}$  term test shows us this series diverges

$$\lim_{n \rightarrow \infty} \left( (-1)^{n+1} \left( \frac{1+2n}{n} \right) \right) \text{ d.n.e. (so its not equal to zero)}$$

The sequence oscillates at  $\infty$  between 2 and -2.

This same logic can be applied whenever you find an alternating series with  $b_n \rightarrow 0$ . They all diverge.

Th<sup>n</sup> = (Ratio Test) We define  $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

(a.) If  $L < 1$  then  $\sum a_n$  is convergent

(b.) If  $L > 1$  or  $L = \infty$  then  $\sum a_n$  is divergent

E16  $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$  does it converge? Well consider

$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{3n^3} \right|$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} \left| \left( \frac{n+1}{n} \right)^3 \right|$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} \left| \left( 1 + \frac{1}{n} \right)^3 \right|$$

$$= \frac{1}{3} < 1 \quad \therefore \boxed{\sum (-1)^n \frac{n^3}{3^n} \text{ converges by Ratio Test}}$$

E17  $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$  Use Ratio Test to investigate convergence of this series,

$$L = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{3} \left( \frac{2^{n+1} + 5}{2^n + 5} \right) \right|$$

$$\stackrel{f}{=} \lim_{n \rightarrow \infty} \left| \frac{1}{3} \left( \frac{\ln(2) 2^{n+1}}{\ln(2) 2^n} \right) \right| \quad (\text{extending } n \text{ to be continuous variable})$$

$$= \frac{2}{3} < 1 \quad \therefore \boxed{\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \text{ converges by Ratio Test.}}$$

Calculate it

$$\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n + \sum_{n=1}^{\infty} \frac{5}{3^n}$$

$$= \frac{2/3}{1 - 2/3} + \frac{5/3}{1 - 1/3} \leftarrow \begin{matrix} \text{geometric series result} \\ \text{applied twice} \end{matrix}$$

$$= \frac{2}{1} + \frac{5}{2} = \boxed{9/2}$$

E18] Show that  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for any  $x$ .

We'll use the ratio test. Here  $a_n = \frac{x^n}{n!}$  so,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| x \cdot \frac{n!}{(n+1)!} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1}{(n+1)n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} \right) \\ &= 0 < 1 \quad \therefore \text{By ratio test } \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges.} \end{aligned}$$

Summary of Tests to determine convergence of  $s = \sum a_n$

1.)  $n^{\text{th}}$  term test: said if  $\lim_{n \rightarrow \infty} (a_n) \neq 0$  then  $s$  diverges.

2.) geometric series: if  $\frac{a_{n+1}}{a_n} = r$  then  $s = \frac{a}{1-r}$  if  $|r| < 1$   
if the ratio  $r \geq 1$  the geometric series diverges.

3.) telescoping series: We could actually calculate  $S_n$  because  
of a bunch of nice cancellations, sometimes needs partial frac.

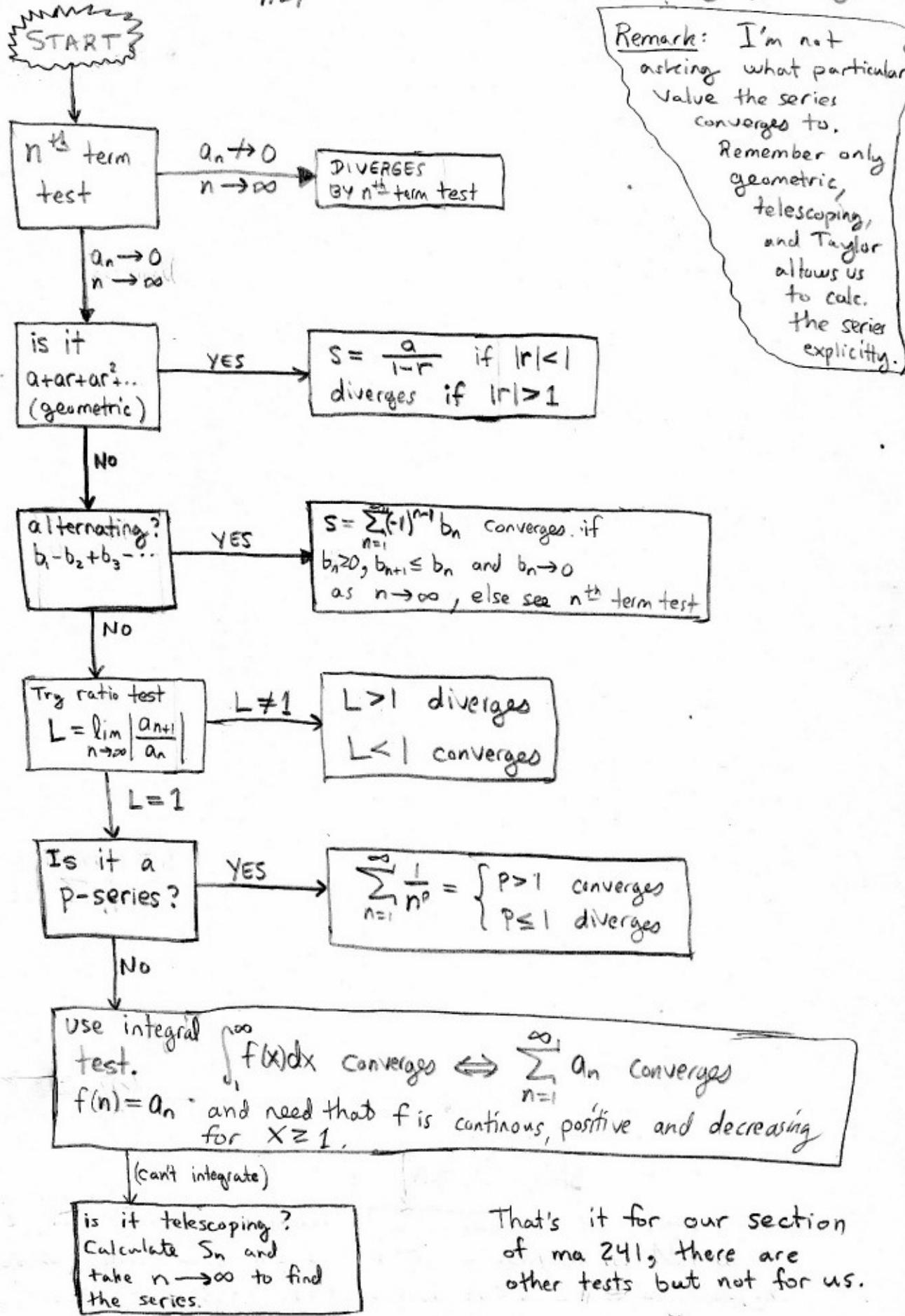
4.) integral test: can use conv/div of improper integrals  
to infer conv/div. of the series.

5.) alternating series test: if series alternates with  
decreasing terms whose limit is zero as  $n \rightarrow \infty$  then  $s$  converges.

6.) ratio test:  $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  then  $L < 1$  means that  
the series converges,  $L > 1$  says it diverges and  $L = 1$   
tells us nothing.

- this list is for my course only! There are many more tests which we don't cover.

Given a series  $S = \sum_{n=1}^{\infty} a_n$  when does it converge/diverge?



That's it for our section of ma 241, there are other tests but not for us.