INTEGRATING FACTOR METHOD

- This topic is in the additional topics category of Stewart, there is a pdf "Linear Differential Equations" which has several pages on this method, also the homework is in that pdf plus sol^n's.

- The integrating factor method will solve just about any DEq^n of the following form (assuming P & Q are continuous)

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)} - (*)$$

we wish to solve $(*)$. Notice that separation of variables doesn't help, there's no way to put this eq^n into the form (stuff in x)(stuff in y) $= \frac{dy}{dx}$. Well, actually there is a way if we use <u>calculus</u>,

E1 Consider $\frac{dy}{dx} + \frac{1}{x}y = 2$. Assume $x > 0$,

1.) calculate $I = \exp\left(\int \frac{1}{x}dx\right) = \exp(\ln|x|) = |x| = x.$

2.) Multiply the DEq^n by the <u>integrating factor</u> I

$$x\frac{dy}{dx} + y = 2x$$

3.) Use the product rule, implicit differention comes into play,

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + \frac{dx}{dx}y = x\frac{dy}{dx} + y = 2x$$

4.) Now we can integrate both sides, use the FTC,

$$\int \frac{d}{dx}(xy)\,dx = \int 2x\,dx \Rightarrow xy = x^2 + C$$

$$\Rightarrow \boxed{y = \frac{x^2+C}{x}}$$

<u>Remark</u>: notice the constant <u>need</u> not stand alone.

We could prove that the method used in $\boxed{E1}$ will solve any <u>linear</u> DEq^{n} of first order but instead we'll continue with more examples,

$\boxed{E2}$ Solve $\dfrac{dy}{dx} = x\sin(2x) + y\tan(x)$ $\quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

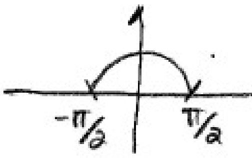
1.) put into the standard form

$$\frac{dy}{dx} - y\tan(x) = x\sin(2x)$$

identify that $P(x) = -\tan(x)$ and $Q(x) = x\sin(2x)$.

2.) Calculate

$$\begin{aligned} I &= \exp\left(\int P(x)\,dx\right) \\ &= \exp\left(-\int \tan(x)\,dx\right) \\ &= \exp\left(-\int \frac{\sin(x)\,dx}{\cos(x)}\right) \\ &= \exp\left(-\int \frac{-du}{u}\right) \\ &= \exp\left(\ln|\cos(x)|\right) \\ &= |\cos(x)| \\ &= \cos(x). \end{aligned}$$



$\cos(x) > 0$ on the interval $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

3.) Multiply DEq^{n} in standard form by $I(x)$

$$\cos(x)\frac{dy}{dx} - y\cos(x)\tan(x) = x\sin(2x)\cos(x)$$

4.) recall $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\sin(2x) = 2\sin(x)\cos(x)$ thus

$$\underbrace{\cos(x)\frac{dy}{dx} - \sin(x)\,y}_{\frac{d}{dx}\left(\cos(x)\,y\right)} = 2x\cos^2(x)\sin(x)$$

$$\frac{d}{dx}\left(\cos(x)\,y\right) = 2x\cos^2(x)\sin(x)$$

do this integral for a Bonus pt.

5.) $\displaystyle\int \frac{d}{dx}\left(\cos(x)\,y\right)dx = \int 2x\cos^2(x)\sin(x)\,dx \quad \therefore \quad \boxed{y = \frac{1}{\cos(x)}\int 2x\cos^2(x)\sin(x)\,dx}$

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E3 Consider $t\ln(t)\dfrac{dr}{dt} + r = te^t$. Here we have independent variable t and dependent variable r.

1.) $$\frac{dr}{dt} + \frac{1}{t\ln(t)}r = \frac{te^t}{t\ln(t)} = \frac{e^t}{\ln(t)}$$ assume $t > 1$.

2.) $$I = \exp\left(\int \frac{1}{\ln(t)}\frac{dt}{t}\right) = \exp\left(\int \frac{du}{u}\right) = \exp\left(\ln|\ln(t)|\right) = |\ln(t)| = \ln(t)$$

$u = \ln(t)$
$du = dt/t$

$(t > 1)$

3.) $$\underbrace{\ln(t)\frac{dr}{dt} + \frac{1}{t}r} = e^t$$: multiplied by I

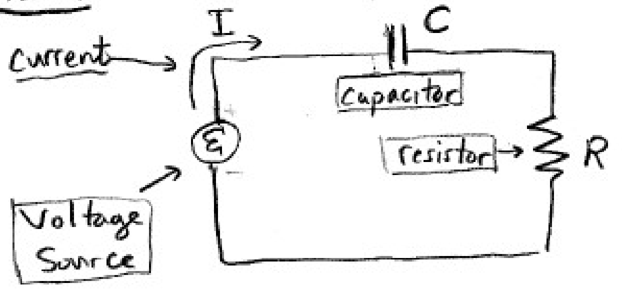
4.) $$\frac{d}{dt}\left(\ln(t)\,r\right) = e^t$$: used product rule

5.) $$\int \frac{d}{dt}\left(\ln(t)r\right)dt = \int e^t dt \Rightarrow r\ln(t) = e^t + c$$

$$\therefore \boxed{r = \frac{e^t + c}{\ln(t)}}$$

E4

Remark: a nice application is RC-circuits. The current I is the rate of change $I = dQ/dt$ where Q is the charge,



$$RI + \frac{Q}{C} = \mathcal{E} \quad \left(\text{Kirchoff's Law}\right)$$

RI: Ohm's Law

$\frac{Q}{C}$: Def$^{\underline{n}}$ of Capacitance. $V_{capacitor} = \frac{Q}{C}$.

Then Kirchoff's Law for this example reveals a 1^{st} order linear ODE in Q,

$$R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}$$

$$\therefore \frac{dQ}{dt} + \frac{1}{RC}Q = \frac{\mathcal{E}}{R}$$

we can solve by

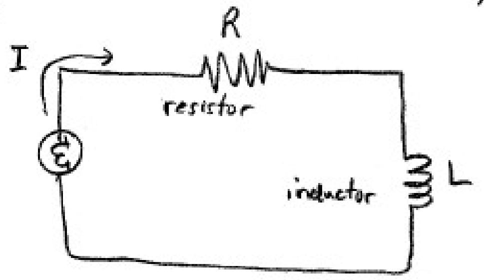
① $I = \exp\left(\int \frac{dt}{RC}\right) = e^{t/RC}$

② $e^{t/RC}\frac{dQ}{dt} + \frac{1}{RC}e^{t/RC}Q = \frac{\mathcal{E}e^{t/RC}}{R}$

③ $\frac{d}{dt}\left(e^{t/RC}Q\right) = \frac{\mathcal{E}e^{t/RC}}{R}$

④ $Q(t) = e^{-t/RC}\int \frac{\mathcal{E}}{R}e^{t/RC}dt$

E5 The other nice application to circuits is the RL-circuit



$$L\frac{dI}{dt} + RI = \mathcal{E}$$

≈ def^n of inductance

Ohm's Law

(Kirchoff's Law says sum of voltages is zero.)

Then $\frac{dI}{dt} + \frac{R}{L} I = \frac{\mathcal{E}}{L}$ so the integrating factor method will solve this. Suppose $\mathcal{E}$ and R and L are constants,

① $\mu = \exp\left(\int \frac{R}{L} dt\right) = \exp(Rt/L)$

② $e^{Rt/L}\frac{dI}{dt} + \frac{R}{L}e^{Rt/L} I = e^{Rt/L}\frac{\mathcal{E}}{L}$

μ is the notation we use in ma 341 for the int. factor

③ $\frac{d}{dt}\left(e^{Rt/L} I\right) = \frac{\mathcal{E}}{L}e^{Rt/L}$

④ $e^{Rt/L} I = \int \frac{\mathcal{E}}{L}e^{Rt/L}dt = \frac{\mathcal{E}}{L}\frac{L}{R}e^{Rt/L} + C_1$

$$\therefore \quad I = \mathcal{E}/R + C_1 e^{-Rt/L}$$

$$\therefore \quad I = \mathcal{E}/R + C_1 e^{-t/\tau}$$

$\tau = L/R$ (time constant for RL-circuit)

General Strategy

① Put into the form $\frac{dy}{dx} + Py = Q$

② Calculate $I = \exp\left(\int P(x)\,dx\right)$

③ Multiply by I

④ Use Product Rule $\frac{d}{dx}(Iy) = IQ$

⑤ Integrate both sides then solve for y.

$$y = \frac{1}{I}\int QI\,dx$$

this is the integrating factor method.