

Non-Homogeneous 2nd order Linear ODE's

We have completely solved $ay'' + by' + cy = 0$ by solving the algebra problem $a\lambda^2 + b\lambda + c = 0$ to get $\lambda_1, \lambda_2 \Rightarrow Y = C_1 Y_1 + C_2 Y_2$ where Y_1, Y_2 must be one of the following,

- I.) $e^{\lambda_1 x}, e^{\lambda_2 x}$
- II.) e^{2x}, xe^{2x}
- III.) $e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x \quad (\lambda_{1,2} = \alpha \pm i\beta)$

These are the "fundamental sol's"

Now we will see how to solve a certain class of eq's with a fairly simple non-homogeneous (or forcing term)

$$ay'' + by' + cy = g(x) \quad \text{our overall goal}$$

The general sol^{1c} will have the form:

in §7.8 is to solve this eqn.

$$Y = Y_c + Y_p$$

$$\begin{array}{ll} \text{Complementary Sol} & \text{Particular Sol} \\ (ay_c'' + by_c' + cy_c = 0) & (ay_p'' + by_p' + cy_p = g(x)) \end{array}$$

$$Y_c = C_1 Y_1 + C_2 Y_2$$

{We already learned this
in the last section}

{We'll find Y_p by the
method of undetermined coefficients}

Method of Undetermined Coefficients

I'll illustrate by example then give general prescription.

E8 $y'' + 9y = \sin(x)$ find general sol^{1c}:

$$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i \therefore Y_c = C_1 \sin(3x) + C_2 \cos(3x)$$

Next guess $Y_p = A \sin(x) + B \cos(x)$

$$Y_p' = A \cos(x) - B \sin(x)$$

$$Y_p'' = -A \sin(x) - B \cos(x) = -Y_p$$

Substitute into our original eqⁿ:

$$Y_p'' + 9Y_p = -Y_p + 9Y_p = 8A \sin(x) + 8B \cos(x) = \sin(x)$$

Thus comparing coeff's $\Rightarrow A = 1/8 \neq B = 0$

$$Y = C_1 \sin(3x) + C_2 \cos(3x) + \frac{1}{8} \sin(x)$$

$$\boxed{E9} \quad y'' + y = \sin(x) \Rightarrow \lambda = \pm i \Rightarrow Y_c = C_1 \cos(x) + C_2 \sin(x)$$

Notice if we try $y_p = A\cos(x) + B\sin(x)$ we'll find $y_p'' + y_p = 0 \neq \sin(x)$, so some other guess must be made for y_p . With II, in mind we'll try to multiply our original "naive guess" by x :

$$y_p = x(A\sin(x) + B\cos(x))$$

$$y_p' = A\sin(x) + B\cos(x) + x(A\cos(x) - B\sin(x))$$

$$y_p'' = A\cos(x) - B\sin(x) + A\cos(x) - B\sin(x) + x(-A\sin(x) - B\cos(x)) \\ - y_p$$

Now

$$y_p'' + y_p = 2A\cos(x) - 2B\sin(x) - y_p + y_p \\ = 2A\cos(x) - 2B\sin(x) = \sin(x)$$

Comparing coefficients: $A = 0 \neq B = -\frac{1}{2}$. So we find

$$Y = C_1 \cos(x) + C_2 \sin(x) - \frac{1}{2}x \cos(x)$$

$$\boxed{E10} \quad y'' + 8y' + 16y = 3x^2 - 2 \quad \boxed{\text{find gen. sol}}$$

$$\lambda^2 + 8\lambda + 16 = (\lambda + 4)(\lambda + 4) \therefore \lambda = -4 \therefore Y_c = C_1 e^{-4x} + C_2 x e^{-4x}$$

Now guess: $y_p = Ax^2 + Bx + C \leftarrow (\text{no overlap}) \uparrow$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Now Subst.

$$y_p'' + 8y_p' + 16y_p = 2A + 8[2Ax + B] + 16[Ax^2 + Bx + C]$$

$$= (2A + 8B + 16C) + x(16A + 16B) + x^2(16A) = 3x^2 - 2$$

Equate Coefficients:

$$x^2: \quad 16A = 3 \quad \therefore A = \frac{3}{16}$$

$$x^1: \quad 16(A+B) = 0 \quad \therefore B = -A = \boxed{-\frac{3}{16} = B}$$

$$x^0: \quad 2A + 8B + 16C = -2 \quad \Rightarrow 16C = -2 + 6A$$

$$C = \frac{-2 + 6A}{16} = \frac{-2 + \frac{18}{16}}{16} = \frac{-7}{128}$$

$$Y = C_1 e^{-4x} + C_2 x e^{-4x} + \frac{3}{16}x^2 - \frac{3}{16}x - \frac{7}{128}$$

$$\boxed{EII} \quad y'' + 9y = e^{2x} \sin(x)$$

$$\lambda^2 + 9 = 0 \therefore \lambda = \pm 3i \therefore y_c = C_1 \sin(3x) + C_2 \cos(3x)$$

$$\text{Guess } y_p = e^{2x}(A \sin(x) + B \cos(x))$$

$$\begin{aligned} y'_p &= 2e^{2x}(A \sin(x) + B \cos(x)) + e^{2x}(A \cos(x) - B \sin(x)) \\ &= e^{2x}[(2A - B) \sin(x) + (2B + A) \cos(x)] \end{aligned}$$

$$\begin{aligned} y''_p &= 2e^{2x}[(2A - B) \sin(x) + (2B + A) \cos(x)] + e^{2x}[(2A - B) \cos(x) - (2B + A) \sin(x)] \\ &= e^{2x}[\sin(x)((4A - 2B) - (2B + A)) + \cos(x)(4B + 2A + 2A - B)] \\ &= e^{2x}[\sin(x)(3A - 4B) + \cos(x)(3B + 4A)] \end{aligned}$$

Subst.

$$\begin{aligned} y'' + 9y_p &= e^{2x}[\sin(x)(3A - 4B) + \cos(x)(3B + 4A)] + 9e^{2x}(A \sin(x) + B \cos(x)) \\ &= e^{2x}[\sin(x)(12A - 4B) + \cos(x)(12B + 4A)] = e^{2x} \sin(x) \end{aligned}$$

Equating Coefficients:

$$12A - 4B = 1$$

$$12B + 4A = 0 \Rightarrow A = -3B$$

$$\Rightarrow 12(-3B) - 4B = 1$$

$$\Rightarrow -40B = 1 \therefore B = -\frac{1}{40} \therefore A = \frac{3}{40}$$

Hence

$$y = C_1 \sin(3x) + C_2 \cos(3x) + e^{2x} \left(\frac{3}{40} \sin(x) - \frac{1}{40} \cos(x) \right)$$

Some conceptual foundations to our recent calculations

(190b)

- Comparing Coefficients: obvious question, how can we do it? why is it valid? I'll illustrate by example, suppose that

$$(A + B + C)x^2 + (B + C)x + C = 6x^2 + 3x + 1 \quad *$$

How to find A, B, C? We compare coefficients and get

$$\underline{x^0} \quad C = 1 \Rightarrow \boxed{C=1}$$

$$\underline{x^1} \quad B + C = 3 \Rightarrow \boxed{B=2}$$

$$\underline{x^2} \quad A + B + C = 6 \Rightarrow \boxed{A=3}$$

But how did I know the eqn's above held given *?

Well, since * holds for all values of x we can choose $x=0$ to find

$$(A + B + C)0^2 + (B + C) \cdot 0 + C = 6(0)^2 + 3(0) + 1 \Rightarrow \boxed{C=1}$$

Ok, now differentiate *,

$$(A + B + C)(2x) + B + C = 12x + 3 \quad *'$$

And evaluate *' at $x=0$ to get

$$(A + B + C)(2 \cdot 0) + B + C = 12(0) + 3 \Rightarrow \boxed{B + C = 3}$$

We've recovered the $\underline{x^0}$ and $\underline{x^1}$ eqn's. Now differentiate *' to obtain

$$(A + B + C) \cdot 2 = 12 \Rightarrow \boxed{A + B + C = 6}$$

So, we see that the fact * holds for all x , and a little differentiation, will validate the eqn's we claimed to be true by "comparing coefficients".

- In fact, it is not hard to see the ideas above can prove equating coefficients for n^{th} degree polynomials. we'd just have to differentiate n -times instead of 2.

More conceptual foundations

190c

I have mentioned the idea of "linear independence" let's define it (for functions of a real variable x)

Def^b $f(x)$ and $g(x)$ are linearly dependent if $f(x) = c g(x)$ for all x in $\text{dom}(f) \cap \text{dom}(g)$. If we cannot write $f(x)$ as a constant multiple of $g(x)$ then we say $f(x)$ and $g(x)$ are linearly independent (L.I.)

Proposition: if $f(x)$ and $g(x)$ are linearly dependent then they have proportional slopes.

Pf/^c just differentiate $f(x) = c g(x) \Rightarrow f'(x) = c g'(x)$.

Examples of Linear Independence/dependence

1.) x is L.I. from x^2 notice $\frac{dx}{dx} = 1$ while $\frac{d}{dx} x^2 = 2x$
clearly 1 is not proportional to $2x$.
thus x cannot be linearly dep. on x^2
hence x & x^2 are L.I.

2.) $c_1 e^x$ and $c_2 e^x$ are linearly dependent $c_1 e^x = \left(\frac{c_1}{c_2}\right) c_2 e^x$.
(assume $c_1, c_2 \neq 0$)

3.) $c_1 e^x$ and $c_2 x e^x$ are L.I.

4.) $1, x, x^2, x^3, \dots, x^n$ are L.I. (pairwise)

5.) $\sin(x)$ and $\cos(x)$ are L.I. (think about the graphs.)

Remark: Given some set of linearly independent functions we can compare coefficients in an eqⁿ involving those functions.

Method of Undetermined Coefficients

Now that we've seen a few examples I'll elaborate on a systematic method to "guess" Y_p . To begin we're given

$$aY'' + bY' + cY = g(x)$$

to find the general solⁿ proceed as follows,

- ① Find Y_c the complementary (aka homogeneous sol^a) this will be a linear combination of two fundamental solⁿ's Y_1 & Y_2 ; $Y_c = C_1 Y_1 + C_2 Y_2$.
- ② Take infinitely many derivatives of $g(x)$ and see what type of functions result. (say $g_1(x), g_2(x), g_3(x), \dots$). Form the naive particular sol^b by taking a linear combination of those functions

$$Y_p(\text{naive}) = A g_1(x) + B g_2(x) + C g_3(x) + \dots$$

Now if g_1, g_2, g_3, \dots are all linearly independent from Y_1 & Y_2 then we say there is no "overlap" and conclude that $Y_p = Y_p(\text{naive})$. Otherwise, if g_1 is linearly dependent to Y_1 (or Y_2) then we modify the particular sol^b to be

$$Y_p = A x g_1(x) + B g_2(x) + C g_3(x) + \dots$$

and this will work provided $x g_1, g_2, g_3, \dots$ are linearly independent from Y_1 & Y_2 , however if $x g_1$ is linearly dependent to Y_2 (it is certainly linearly indep. from Y_1) then we must multiply by x again and use

$$Y_p = A x^2 g_1(x) + B g_2(x) + C g_3(x) + \dots$$

If g_1 and g_2 are linearly dependent to Y_1 & Y_2 then we must modify both g_1 & g_2 as follows,

$$Y_p = A x g_1(x) + B x g_2(x) + C g_3(x) + \dots$$

For 2nd order ODEg^a's that should do it. The overriding goal is that all the funcs composing Y_c & Y_p should be linearly independent.

③ So we've "guessed" y_p now take two derivatives, that is calculate y_p' and y_p'' .

④ Substitute y_p, y_p', y_p'' into $a y_p'' + b y_p' + c y_p = g(x)$

⑤ Algebraically combine like terms then compare coefficients of like terms on the LHS & RHS to obtain equations between A, B, C, ...

⑥ Solve the system of linear equations that results from ⑤, you'll find specific numeric values for A, B, C, ... (btw if you did ② incorrectly it will become glaringly obvious as the eq's will give nonsensical results like $3=0$ and so on...)

⑦ We construct the general solⁿ,

$$Y_g = c_1 Y_1 + c_2 Y_2 + y_p$$

the only coefficients left unknown in the end should be c_1 & c_2 . We can only determine c_1 & c_2 if we have some additional data about the system (either initial or boundary conditions). You should use Y_g to satisfy initial conditions if they're given, it is Y_g that is the solⁿ to the DEqⁿ (not Y_c or y_p alone!)

A few examples about overlap and forming the "naive" guess.

I do not claim, or intend, for this list to be comprehensive. My goal here is for you to understand the general comments made on 190d → 190e

$g(x)$	naive Y_p	possible modifications for Y_p if overlap.	
1. $\sin(x) \cosh(x)$	$(A\cos(x)+B\sin(x))(C\cosh(x)+D\sinh(x))$	no comment.	
2. 2^x	$A \cdot 2^x$	$Ax 2^x$	$\{ Y_c = e^{2\ln(2)x} = e^{\ln(2^x)} = 2^x \}$
3. x^5	$Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$	$x(Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F)$	
4. $e^x + \sin(x) + x^2$	$Ae^x + B\sin(x) + C\cos(x) + Ex^2 + Dx + E$	no comment.	
5. $e^x \sin(x) x^2$	$e^x (A\sin(x) + B\cos(x)) (Cx^2 + Dx + E)$	no comment.	
6. 3	A	Ax	
7. $3x^2$	$Ax^2 + Bx + C$	$x(Ax^2 + Bx + C)$	
8. $\sin(2x)$	$A\sin(2x) + B\cos(2x)$	$x(A\sin(2x) + B\cos(2x))$	
9. $e^x \sin(x)$	$e^x (A\sin(x) + B\cos(x))$	$xe^x (A\sin(x) + B\cos(x))$	
10. $t e^{et}$	$At e^{et} + Be^{et}$	$At^2 e^{et} + Bt e^{et}$	$At^3 e^{et} + Bt^2 e^{et}$ $\{ Y_c = c_1 e^{et} + c_2 t e^{et} \}$
11. $3x$	$Ax + B$	$x^2 (Ax + B)$	
12. $\cosh(x)$	$A\cosh(x) + B\sinh(x)$	$x (A\cosh(x) + B\sinh(x))$	
13. $e^{4t} + e^t$	$Ae^{4t} + Be^t$	$At e^{4t} + Be^t$ $\{ Y_c = c_1 e^{4t} + c_2 e^{3t} \}$	$t(Ae^{4t} + Be^t)$ $\{ Y_c = c_1 e^{3t} + c_2 e^{2t} \}$
Philosophy of Guess: Y_c must be linearly independent of Y_p .			

Additional Examples on choosing Y_p

190g

3. a.) $y'' + 2y' + 10y = x^2 + x\cos(x)$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = -\frac{2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$$

$$\Rightarrow Y_c = e^{-x}(C_1 \cos(3x) + C_2 \sin(3x))$$

then we guess

$$Y_p = Ax^2 + Bx + C + x(D\cos(x) + E\sin(x)) + F\cos(x) + G\sin(x)$$

no overlap so that'll do.

b.) $y'' + 2y' + y = e^{-x}$

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \quad \therefore \lambda = -1 \text{ twice}$$

$$\Rightarrow Y_c = C_1 e^{-x} + C_2 x e^{-x}$$

naively $Y_p = Ae^{-x}$ (overlaps)

less naive $Y_p = Axe^{-x}$ (still overlaps)

correct $Y_p = Ax^2 e^{-x}$

c.) $y'' + 9y = \cos(3x) - 6$

$$\lambda^2 + 9 = 0 \quad \therefore \lambda = \pm 3i \Rightarrow Y_c = C_1 \cos(3x) + C_2 \sin(3x)$$

$Y_p^n = A\cos(3x) + B\sin(3x) + C$, naive, it overlaps Y_c .

$$Y_p = x(A\cos(3x) + B\sin(3x)) + C$$

Notice Cx will not work.

d.) $y'' + 8y' + 12y = e^{-2x} + 7x$

$$\lambda^2 + 8\lambda + 12 = (\lambda + 2)(\lambda + 6) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -6$$

$$\Rightarrow Y_c = C_1 e^{-2x} + C_2 e^{-6x}$$

$Y_p = Axe^{-2x} + Bx + C$ has no overlap, it'll work.

e.) $y'' + 16y = xe^x \sin(x)$

$$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i \Rightarrow Y_c = C_1 \cos(4x) + C_2 \sin(4x)$$

think about differentiating $xe^x \sin(x)$. The x can go to 1 but the e^x survives and $\sin(x)$ becomes $\cos(x)$ thus

$$Y_p = e^x(A\sin(x) + B\cos(x) + x(C\sin(x) + D\cos(x)))$$

clearly no overlap here.

Additional Examples

$$\textcircled{2} \quad Y'' + 5Y' + 6Y = 12e^x + 6x + 11$$

to begin find Y_c .

$$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \quad \therefore \lambda_1 = -3, \lambda_2 = -2$$

$$\Rightarrow Y_c = C_1 e^{-3x} + C_2 e^{-2x}$$

now we find the particular solⁿ using the method of undetermined coefficients. Begin with the educated guess

$$Y_p = Ae^x + Bx + C$$

$$Y_p' = Ae^x + B$$

$$Y_p'' = Ae^x$$

$$Y_p'' + 5Y_p' + 6Y_p = 12e^x + 6x + 11$$

$$Ae^x + 5(Ae^x + B) + 6(Ae^x + Bx + C) = 12e^x + 6x + 11$$

$$e^x(A + 5A + 6A) + x(6B) + 5B + 6C = e^x(12) + x(6) + 11$$

Equate Coefficients of e^x , x and constants,

$$e^x: 12A = 12 \quad \Rightarrow \boxed{A=1}$$

$$x: 6B = 6 \quad \Rightarrow \boxed{B=1}$$

$$x^0: 5B + 6C = 11 \quad \Rightarrow 6C = 11 - 5 = 6 \quad \Rightarrow \boxed{C=1}$$

So we find the general solⁿ $Y_g = Y_c + Y_p$ is

$$\boxed{Y_g = C_1 e^{-3x} + C_2 e^{-2x} + e^x + x + 1}$$

Why you don't want a take-home

$$(2) \quad Y'' - 2Y' + Y = e^x + x\cos(2x) + x^4$$

$$\text{i.) } \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \Rightarrow Y_c = C_1 e^x + C_2 x e^x$$

$$\text{ii.) } Y_p = \underbrace{Ax^2 e^x}_{Y_{P_1}} + \underbrace{x(B\cos(2x) + C\sin(2x))}_{Y_{P_2}} + \underbrace{D\cos(2x) + E\sin(2x)}_{Y_{P_3}} + Fx^4 + Gx^3 + Hx^2$$

$$Y_p' = A(2x + x^2)e^x$$

$$Y_p'' = A(2 + 2x + 2x + x^2)e^x$$

$$\begin{aligned} Y_p'' - 2Y_p' + Y_p &= A(x^2 + (2x + x^2)(-2) + 2 + 4x + x^2)e^x \\ &= A(x^2 - 4x - 2x^2 + 2 + 4x + x^2)e^x \\ &= A(2)e^x = e^x \Rightarrow A = \frac{1}{2} \end{aligned}$$

$$Y_{P_2}' = B\cos(2x) + C\sin(2x) + x(-2B\sin(2x) + 2C\cos(2x)) - \Rightarrow \\ C - 2D\sin(2x) + 2E\cos(2x)$$

$$Y_{P_2}' = \cos(2x)[B + 2Cx + 2E] + \sin(2x)[C - 2Bx - 2D]$$

$$\begin{aligned} Y_{P_2}'' &= -2\sin(2x)[B + 2Cx + 2E] + \cos(2x)[2C] \\ &\quad + 2\cos(2x)[C - 2Bx - 2D] + \sin(2x)[-2B] \end{aligned}$$

$$Y_{P_2}'' = \cos(2x)[2C + 2C - 4Bx - 4D] + \sin(2x)[-2B - 4Cx - 4E - 2B]$$

$$\begin{aligned} x\cos(2x) &= Y_p'' - 2Y_p' + Y_p = \cos(2x)[4C - 4D - 4Bx] + \sin(2x)[-4E - 4B - 4Cx] \Rightarrow \\ &\quad C - 2\cos(2x)[B + 2E + 2Cx] - 2\sin(2x)[C - 2D - 2Bx] \Rightarrow \\ &\quad C\cos(2x)[Bx + D] + \sin(2x)[Cx + E] \end{aligned}$$

$$\underline{\cos(2x)} \quad 4C - 4D - 4Bx - 2B - 4E - 4Cx + Bx + D = X$$

$$X(-4B - 4C + B) + 4C - 3D - 2B - 4E = X$$

$$\underline{-3B - 4C = 1} \quad \underline{4C - 3D - 2B - 4E = 0}$$

$$\underline{\sin(2x)} \quad -4E - 4B - 4Cx - 2C + 4D + 4Bx + Cx + E = 0$$

$$X(-4C + 4B + C) + (-4E + E - 4B - 2C + 4D) = 0$$

$$\underline{-3C + 4B = 0} \quad \underline{-3E - 4B - 2C + 4D = 0}$$

$$(2) \quad \left. \begin{array}{l} -3B - 4C = 1 \\ 4C - 3D - 2B - 4E = 0 \\ -3C + 4B = 0 \\ -3E - 4B - 2C + 4D = 0 \end{array} \right\} \sim \left[\begin{array}{cccc|c} -3 & -4 & 0 & 0 & 1 \\ -2 & 4 & -3 & -4 & 0 \\ 4 & -3 & 0 & 0 & 0 \\ -4 & -2 & 4 & -3 & 0 \end{array} \right]$$

augmented
coefficient
matrix

enter the matrix above into TI and use "rref" to find,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3/25 \\ 0 & 1 & 0 & 0 & -4/25 \\ 0 & 0 & 1 & 0 & -22/125 \\ 0 & 0 & 0 & 1 & 4/125 \end{array} \right] \Leftrightarrow \boxed{\begin{array}{l} B = -3/25 \\ C = -4/25 \\ D = -22/125 \\ E = 4/125 \end{array}}$$

Next,

$$Y_p = Fx^4 + Gx^3 + Hx^2 + Ix + J$$

$$8^3 Y_p' = 4Fx^3 + 3Gx^2 + 2xH + I$$

$$Y_p'' = 12Fx^2 + 6Gx + 2H$$

$$\begin{aligned} X^4 &= Y_p''' - 2Y_p' + Y_p = 12Fx^2 + 6Gx + 2H \\ &\quad - 8Fx^3 - 6Gx^2 - 4Hx - 2I \\ &\quad + Fx^4 + Gx^3 + Hx^2 + Ix + J \\ &= 2H - 2I + J \\ &\quad + x(6G - 4H + I) \\ &\quad + x^2(12F - 6G + H) \\ &\quad + x^3(-8F + G) \\ &\quad + x^4(F) \end{aligned}$$

Equating Coefficients:

$$F = 1$$

$$-8 + G = 0 \therefore G = 8$$

$$12 - 48 + H = 0 \therefore H = 36$$

$$48 - 144 + I = 0 \therefore I = 96$$

$$72 - 192 + J = 0 \therefore J = 120$$

$$Y = c_1 e^x + c_2 x e^x + \frac{1}{2} x^2 e^x + x \left(\frac{-3}{25} \cos(2x) - \frac{4}{25} \sin(2x) \right) - \frac{22}{125} \cos(2x) + \frac{4}{125} \sin(2x) + 2$$

$$\hookrightarrow x^4 + 8x^3 + 36x^2 + 96x + 120$$

Carefully I was incorrect about how to set up x^4 , needed $I \neq J$.

Why $y_g = y_c + y_p$ solves $ay'' + by' + cy = g(x)$

(190k)

The complementary solⁿ y_c (aka homogeneous solⁿ) has

$$a y_c'' + b y_c' + c y_c = 0 \quad *$$

Whereas the particular solⁿ y_p satisfies,

$$a y_p'' + b y_p' + c y_p = g(x) \quad **$$

We claimed that $y_c + y_p$ is the general solⁿ, let's prove it. Remember $(f+g)' = f' + g'$ and $(cf)' = cf'$,

$$a(y_c + y_p)'' + b(y_c + y_p)' + c(y_c + y_p) = \dots$$

$$\leftarrow a(y_c'' + y_p'') + b(y_c' + y_p') + c y_c + c y_p$$

$$= \underbrace{a y_c'' + b y_c' + c y_c}_* + \underbrace{a y_p'' + b y_p' + c y_p}_{**}$$

$$= 0 + g(x)$$

$$= g(x) \quad \text{which proves our claim, } y_g = y_c + y_p \text{ solves } ay'' + by' + cy = g(x).$$

Remark: You can think of y_c as being necessary to encode initial conditions into the general solⁿ. Remember a n^{th} order ODE^{gⁿ will have n -arbitrary constants c_1, c_2, \dots, c_n in the general solⁿ. This corresponds to the fact that we need n -independent pieces of data to specify a solⁿ. We've seen this for $n=1$ and $n=2$ if you think about it.}