

# PARTIAL FRACTIONS

The technique of partial fractions will allow us to integrate any rational function! To begin let's see how it works for a fairly simple example then we'll go on to develop the general idea on (26) → (31).

**EO** How to integrate  $\frac{x+5}{x^2+5x+6}$ ? Upto now none of the previous techniques seem to help with this one. So notice that we can break up this fraction into two fractions, following from the factorization of  $x^2+5x+6 = (x+2)(x+3)$ ,

$$\frac{x+5}{(x+2)(x+3)} = \frac{A}{x+3} + \frac{B}{x+2} \quad \leftarrow \text{this guess is verified by algebra below}$$

We can figure out the now unknown A & B as follows, first multiply by denominator to obtain,

$$(x+2)(x+3) \frac{x+5}{(x+2)(x+3)} = (x+2)(x+3) \left[ \frac{A}{x+3} + \frac{B}{x+2} \right]$$

Which gives the (simple to solve) eq<sup>n</sup>,

$$x+5 = (x+2)A + (x+3)B$$

So plug in the roots  $x = -2$  then  $x = -3$  to get

$$-2+5 = 3 = \cancel{(-2+2)}A + (-2+3)B = B \quad \therefore \boxed{B=3}$$

$$-3+5 = 2 = (-3+2)A + \cancel{(-3+3)}B = -A \quad \therefore \boxed{A=-2}$$

Thus we have by the algebra above,

$$\frac{x+5}{(x+2)(x+3)} = \frac{-2}{x+3} + \frac{3}{x+2}$$

Now we can integrate the RHS of the above, remember how?

$$\int \frac{x+5}{(x+2)(x+3)} dx = -2 \int \frac{1}{x+3} dx + 3 \int \frac{1}{x+2} dx$$

$$= -2 \int \frac{du}{u} + 3 \int \frac{dw}{w} \quad \begin{matrix} u = x+3 \\ w = x+2 \end{matrix}$$

$$= \boxed{-2 \ln|x+3| + 3 \ln|x+2| + C}$$

Appendix G § 5.7: PARTIAL FRACTIONS GENERAL SET-UP

The question is how to integrate any rational function  $f(x) = \frac{P(x)}{Q(x)}$   
 This reduces by polynomial long division to the problem of integrating

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

Where  $R(x)$  is the remainder so  $\deg(R(x)) < \deg(Q(x))$ . So how to integrate  $R(x)/Q(x)$  in general? (Note  $S(x)$  is easy! it's a polynomial we can integrate those, no problem). Then from algebra we know that any POLYNOMIAL  $f(x)$  can be factored to:

$$f(x) = (\text{linear factors})(\text{irred. quad. factors})$$

Hence  $R(x) \notin Q(x)$  can be factored likewise, (irred. means irreducible)

$$\frac{R(x)}{Q(x)} = \frac{(\text{linear factors})(\text{irred. quad. factors})}{(\text{linear factors})(\text{irred. quad. factors})}$$

$$= \frac{A}{x-r_1} + \frac{B}{(x-r_1)^2} + \dots + \frac{Cx+D}{x^2+bx+c} + \frac{Ex+F}{(x^2+bx+c)^2} + \dots$$

which and how many of these "Basic Rational" terms depends on the details of  $R(x) \notin Q(x)$ .

Remark: If we can integrate each of the facts appearing in (\*) then we can integrate ANY rational function

## Integrating "Basic" rational functions

I.) Reciprocal of a linear factor. Its a simple u-substitution,  $u = x+a$

$$\int \frac{dx}{x+a} = \int \frac{du}{u} = \ln |u| + C = \boxed{\ln |x+a| + C}$$

II.) Reciprocal of a repeated linear factor. Again a u-subst.  $u = x+a$

$$\int \frac{dx}{(x+a)^n} = \int \frac{du}{u^n} = \frac{1}{n+1} \frac{1}{u^{n-1}} + C = \boxed{\frac{1}{n+1} \frac{1}{(x+a)^{n-1}} + C, n \neq -1}$$

III.a) Reciprocal of an irreducible quadratic ( $b^2 - 4c < 0$  the "discriminant")

$$\int \frac{dx}{x^2+bx+c} = \int \frac{1}{(x+\frac{b}{2})^2 + c - \frac{b^2}{4}} dx \quad : \text{Completing the square in the denominator.}$$

$$= \int \frac{1}{\frac{\alpha^2}{\alpha^2} (x+\frac{b}{2})^2 + \alpha^2} dx \quad : \alpha^2 \equiv c - \frac{b^2}{4}$$

$$= \frac{1}{\alpha^2} \int \frac{1}{(\frac{1}{\alpha}(x+\frac{b}{2}))^2 + 1} dx \quad : \text{factoring out the } \alpha^2$$

$$= \frac{1}{\alpha^2} \int \frac{1}{u^2 + 1} \alpha du$$

$$\left\{ \begin{aligned} u &= \frac{1}{\alpha} (x + \frac{b}{2}) \\ du &= \frac{1}{\alpha} dx \end{aligned} \right.$$

$$= \frac{1}{\alpha} \tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{\sqrt{c - \frac{b^2}{4}}} \tan^{-1}\left(\frac{1}{\sqrt{c - \frac{b^2}{4}}} (x + \frac{b}{2})\right) + C}$$

III.b) X over an irreducible quadratic ( $b^2 - 4c < 0$ )

$$\int \frac{x}{x^2+bx+c} dx = \int \frac{x dx}{(x+\frac{b}{2})^2 + \alpha^2}$$

$$= \int \frac{(x+\frac{b}{2}) dx}{(x+\frac{b}{2})^2 + \alpha^2} - \int \frac{\frac{b}{2} dx}{(x+\frac{b}{2})^2 + \alpha^2} \quad : \text{Sneaky step.}$$

$$= \left(\frac{1}{2} \int \frac{dw}{w}\right) - \frac{b}{2} \frac{1}{\alpha} \tan^{-1}(u) + C \quad \leftarrow \text{using III.a} \quad \left\{ \begin{aligned} w &= (x+\frac{b}{2})^2 + \alpha^2 \\ dw &= 2(x+\frac{b}{2}) dx \end{aligned} \right.$$

$$= \frac{1}{2} \ln(w) - \frac{b}{2} \frac{1}{\alpha} \tan^{-1}(u) + C$$

$$= \boxed{\frac{1}{2} \ln(x^2+bx+c) - \frac{b}{2} \frac{1}{\sqrt{c - \frac{b^2}{4}}} \tan^{-1}\left(\frac{1}{\sqrt{c - \frac{b^2}{4}}} (x + \frac{b}{2})\right) + C}$$



## PARTIAL FRACTIONS:

Goal: given some rational function  $f(x) = \frac{P(x)}{Q(x)}$  integrate it.

Outline: Long  $\div$ , Partial Fractional Decompose it, integrate the "Basic" rational functions that result.

**E1**  $f(x) = \frac{x^4 + x^3 + 2x^2 + 3x - 2}{x^2 - 3x + 2}$   $\leftarrow$  need to do long  $\div$ .

$$\begin{array}{r} x^2 + 4x + 12 \\ x^2 - 3x + 2 \overline{) x^4 + x^3 + 2x^2 + 3x - 2} \\ \underline{-(x^4 - 3x^3 + 2x^2)} \\ 4x^3 + 0x^2 + 3x - 2 \\ \underline{-(4x^3 - 12x^2 + 8x)} \\ 12x^2 - 5x - 2 \\ \underline{-(12x^2 - 36x + 24)} \\ 0 \quad \boxed{31x - 26} \text{ remainder} \end{array}$$

$$\therefore f(x) = \underbrace{x^2 + 4x + 12} + \frac{31x - 26}{x^2 - 3x + 2}$$

can integrate this no problem

we'll use partial fractions to break it up.

$$\frac{31x - 26}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$\leftarrow$  Partial Fractional Decomp. for distinct linear factors.

$$\therefore 31x - 26 = A(x-1) + B(x-2) \quad ; \text{ Multiplying by } (x-2)(x-1)$$

$$\underline{x=1} \quad 5 = -B$$

$$\underline{x=2} \quad 36 = A$$

$$f(x) = x^2 + 4x + 12 + \frac{36}{x-2} - \frac{5}{x-1}$$

$$\int f(x) dx = \frac{x^3}{3} + 2x^2 + 12x + 36 \ln|x-2| - 5 \ln|x-1| + C$$

Once we break  $f(x)$  into "Basic" rational functions we can easily integrate it, using I, II, III, IV we developed. Well I don't expect you to memorize I, II, III, IV, I expect you to reproduce those calculations for particular examples.

$$\boxed{E2} \quad f(x) = \frac{x^2 + 2x + 3}{x^2 + x} = \frac{x^2 + 2x + 3}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 + 2x + 3 = A(x^2 + 1) + (Bx + C)(x)$$

$$\underline{x=0} \quad \boxed{3 = A}$$

$$\underline{x=1} \quad 6 = 2A + B + C \Rightarrow 6 = 6 + B + C \Rightarrow \underline{B = -C}$$

$$\underline{x=2} \quad 11 = 5A + 4B + 2C \Rightarrow 11 = 15 - 4C + 2C \Rightarrow 2C = 4 \Rightarrow C = 2$$

$$\therefore A = 3, B = -2, C = 2$$

$$\int f(x) dx = \int \frac{3}{x} dx + \int \frac{-2}{x^2 + 1} dx - \int \frac{2x}{x^2 + 1} dx$$

$$= 3 \ln|x| + 2 \tan^{-1}(x) - \int \frac{du}{u} \quad ; \quad \left\{ \begin{array}{l} u = x^2 + 1 \\ du = 2x dx \end{array} \right.$$

$$= \boxed{3 \ln|x| + 2 \tan^{-1}(x) - \ln|x^2 + 1| + C}$$

$$\boxed{E3} \quad f(x) = \frac{2x^2 - 3}{(x+1)(x^2 - 1)} = \frac{2x^2 - 3}{(x+1)(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

$$2x^2 - 3 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

$$\underline{x=-1} \quad -1 = -2B \quad \therefore \boxed{B = 1/2}$$

$$\underline{x=1} \quad -1 = 4C \quad \therefore \boxed{C = -1/4}$$

$$\underline{x=0} \quad -3 = -A - B + C = -A - 3/4 \Rightarrow -12 = -4A - 3$$

$$\Rightarrow -9 = -4A$$

$$\Rightarrow \boxed{A = 9/4}$$

$$\int f(x) dx = \frac{9}{4} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{(x+1)^2}$$

$$= \frac{9}{4} \int \frac{du}{u} - \frac{1}{4} \int \frac{dw}{w} + \frac{1}{2} \int \frac{du}{u^2} \quad \left\{ \begin{array}{l} u = x+1 \quad du = dx \\ w = x-1 \quad dw = dx \end{array} \right.$$

$$= \frac{9}{4} \ln|u| - \frac{1}{4} \ln|w| - \frac{1}{2} \frac{1}{u} + C$$

$$= \boxed{\frac{9}{4} \ln|x+1| - \frac{1}{4} \ln|x-1| - \frac{1}{2} \frac{1}{x+1} + C}$$

# Still more PARTIAL FRACTIONS

$$\begin{aligned}
 \text{E4)} \quad f(x) &= \frac{1}{(x+1)(x-2)^2(x^2+1)(x^2+2)^2(x^2+3)^3} \\
 &= \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+1} + 2 \\
 &\quad + \frac{Fx+G}{x^2+2} + \frac{Hx+I}{(x^2+2)^2} + \frac{Jx+K}{x^2+3} + \frac{Lx+M}{(x^2+3)^2} + \frac{Nx+O}{(x^2+3)^3}
 \end{aligned}$$

I'm not going to find A, B, ..., N, O for you but it should be clear how you would go about it.

$$\begin{aligned}
 \text{E5)} \quad f(x) &= \frac{x^2+2x-7}{x(x-3)^5} \\
 &= \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{(x-4)^4} + \frac{E}{(x-5)^5}
 \end{aligned}$$

$$\text{E6)} \quad f(x) = \frac{x^2+3}{(x-2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2}$$

NO must have  $\text{deg}(\text{numerator}) < \text{deg}(\text{denom})$

First we need to do long %.

$$\begin{array}{r}
 x^2 - 4x + 4 \overline{) x^2 + 0x + 3} \\
 \underline{-(x^2 - 4x + 4)} \\
 4x - 1
 \end{array}
 \Rightarrow \frac{x^2+3}{(x-2)^2} = 1 + \frac{4x-1}{(x-2)^2}$$

Now we can do partial fractions on  $\frac{4x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$$4x-1 = A(x-2) + B$$

$$x=0 \Rightarrow -1 = -2A + B$$

$$x=2 \Rightarrow \boxed{7 = B} \Rightarrow 2A = 7+1 \therefore \boxed{A = 4}$$

$$\begin{aligned}
 \int f(x) dx &= \int \left( 1 + \frac{4}{x-2} + \frac{7}{(x-2)^2} \right) dx \\
 &= x + 4 \int \frac{1}{u} du + 7 \int \frac{1}{u^2} du \\
 &= x + 4 \ln|u| - 7 \frac{1}{u} + C \\
 &= \boxed{x + 4 \ln|x-2| - \frac{7}{x-2} + C}
 \end{aligned}$$

$u = x-2$   
 $du = dx$