

In this section we'll apply the basic tenets of algebra-based physics so let's review quickly:

$$m = \text{mass} \quad \{m\} = \text{kg}$$

$$a = \text{acceleration} \quad \{a\} = \text{m/s}^2$$

$$F = \text{force} \quad \{F\} = \text{N} = \text{kg m/s}^2$$

$$P = \text{pressure} \quad \{P\} = \text{N/m}^2 = \text{Pa}$$

$$W = \text{work} \quad \{W\} = \text{N}\cdot\text{m} = \text{J}$$

$$\rho = \text{density} \quad \{\rho\} = \text{kg/m}^3 \quad (\text{taken constant unless otherwise stated})$$

$$A = \text{area} \quad \{A\} = \text{m}^2$$

$$x = \text{distance} \quad \{x\} = \text{m}$$

These are the basic objects of physical interest. In addition we have some laws that relate these variables:

$$F = ma \quad (\text{Newton's second law, applies to objects in straight-line motion only})$$

$$P = \frac{F}{A} \quad (\text{definition of pressure, need } A \text{ to be small enough so } F \text{ is constant, think about it.})$$

$$W = Fx \quad (\text{Work done by } F \text{ displacing distance } x, \text{ need } F \text{ to be constant along displacement.})$$

$$P = \rho gd \quad (\text{Pressure from height } d \text{ of liquid or gas with density } \rho) \\ \text{Sometimes } \delta \text{ is used } \delta = \rho g, \delta = \text{"specific gravity"}$$

$$F = kx \quad (\text{Force from spring with spring constant } k \text{ and } x \text{ is the length stretched/compressed from the equilibrium length.})$$

### Overview:

In this section we'll explore how to extend these average laws of physics by applying them infinitesimally then integrating. We can use calculus to find

- 1.) Work done by varying force (Springs for example)
- 2.) Work required to lift some non-trivial mass (<sup>rope</sup><sub>water tank</sub> etc...)
- 3.) Force from varying pressure (force against dam.)
- 4.) Moments of Inertia & Center of Mass

EI Find magnitude of work done by a spring with  $k = 10 \text{ N/m}$  on a ball attached to the spring as the spring stretches 10 cm from its equilibrium length.



What's the magnitude of work done on the ball?

Hooke's Law says  $F = kx$  so then what is work done during a displacement  $dx$ ? It is  $dW$  which is simply  $dW = Fdx = kx dx$ . Over this tiny displacement the force is constant so the work done is simply force  $\cdot$  distance  $= kx dx$ . Now we add up all the little bits of work done to find the total work,

$$\begin{aligned} W &= \int_0^{0.1} 10x dx \\ &= 5x^2 \Big|_0^{0.1} \\ &= 0.05 \text{ J} \end{aligned}$$

Philosophy of units: Use Nm, kg, s then put units in when we get to the answer. It was important to use 0.1m rather than 10cm in this calculation. I don't require you to demonstrate how the units cancel, we'll leave that to PY 205. I do expect correct units on your answer.

**E2** Given any force which is a function of distance  
 X what is the magnitude of work done by the  
 force as it displaces an object from  $x=a$  to  $x=b$

Call the force  $F = F(x)$  then during the displacement  $dx$  we can deduce work  $dW$  is done where

$$dW = F dx$$

Thus the total work is simply,

$$W = \int dW = \int_a^b F dx$$

Work is the integral of the force.

Remark: We assume 1-dimensional motion. For more dimensions we need the math of vectors, we'll get to that in calc. III.

**E3** Given that 2J of work is required to stretch a spring 12 cm from its equilibrium length how far will the same spring stretch when a 30N force is applied to it? (#7 §6.5 pg. 479)

$$W = \int_0^{0.12} kx dx = \frac{1}{2} kx^2 \Big|_0^{0.12} = 0.0072k = 2$$

So we deduce  $k = 277.8$ . Remember Hooke's Law

$$F = kx = 30$$

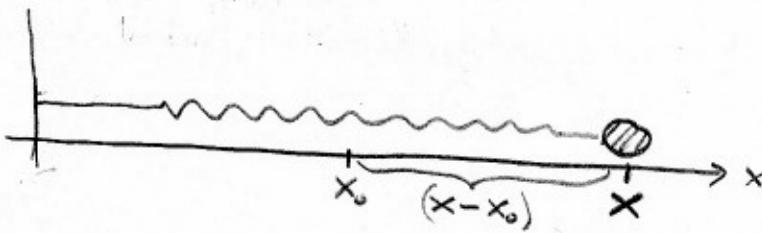
$$\text{Thus } x = \frac{30}{k} = \frac{30}{277.8} = 0.1080 \text{ m. That}$$

is the spring stretches 10.8cm when 30N is applied.

E4 Suppose that 6 J<sub>lbs</sub> of work are required to stretch a spring from 10m to 12m and another 10 J<sub>lbs</sub> are required to stretch it an additional 2m. What is the natural length of the spring?

Lets give the unknown natural length a name :  $x_0$

$$F = k(x - x_0)$$



its convenient  
to change the  
meaning of "x"  
for this problem.

Now lets use what we know;  $dW = Fdx = k(x - x_0)dx$ ,

$$\begin{aligned} 6 &= W_{10 \rightarrow 12} = \int_{10}^{12} k(x - x_0)dx \\ &= \left( \frac{1}{2}kx^2 - kx_0x \right) \Big|_{10}^{12} \\ &= \frac{1}{2}k(144 - 100) - kx_0(12 - 10) \\ &= \underline{22k - 2x_0k = 6} \end{aligned}$$

$$\begin{aligned} 10 &= W_{12 \rightarrow 14} = \int_{12}^{14} k(x - x_0)dx \\ &= \left( \frac{1}{2}kx^2 - kx_0x \right) \Big|_{12}^{14} \\ &= \frac{1}{2}k(196 - 144) - kx_0(14 - 12) \\ &= \underline{26k - 2x_0k = 10} \end{aligned}$$

I'll let you finish this example, what's  $x_0$ ?

(hmm, this does seem similar to a hwk. problem...)

(I've changed the units from the textbook problem.)

E5) Find magnitude of work required to lift a 10m cable with a uniformly distributed mass of 10kg.

Lets define  $l$  to be the length of the cable being lifted, this will vary from  $l = 10\text{ m} \rightarrow l = 0\text{ m}$  as the cable is lifted.

We should find the work  $dW$  done as we lift length  $l$  a distance  $dl$ ,

$$dW = F dl$$

$$= -mg dl$$

$$= -\lambda g dl$$

$$\lambda = \frac{\text{mass}}{\text{length}} = \frac{10\text{ kg}}{10\text{ m}} = 1 \text{ kg/m}$$

" $\lambda$  is linear mass density"

$$m = \lambda l$$

Now add up the work

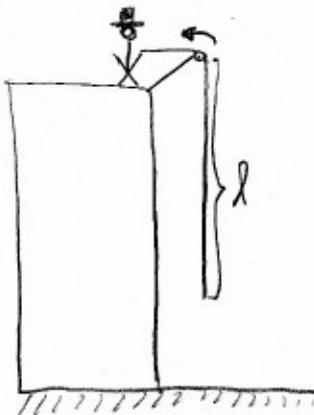
$$W = \int_{10}^0 -\lambda g l dl$$

$$= -\lambda g \left( \frac{1}{2} l^2 \right) \Big|_0^0$$

$$= \lambda g \frac{1}{2} (10)^2$$

$$= -1 \cdot 9.8 \cdot 50$$

$$= \boxed{490 \text{ J}}$$



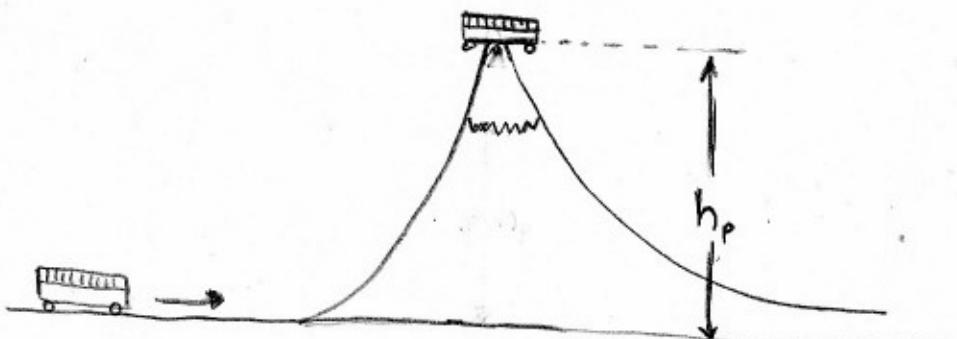
E6) Imagine a bus has a route which travels up a mountain. As the bus travels up the mountain it drops off passengers causing its mass to vary according to  $m = \left(3 - \frac{y}{y+2}\right)m_0$

Find magnitude of work done against gravity by the bus as it travels its route from  $y=0$  all the way upto  $y=h_{\text{peak}} = h_p$

Notice  $F = mg$  is not constant as  $m$  varies with  $y$ . We need calculus to solve it! So find  $dW$  from the bus moving  $dy$  against gravity

$$dW = mg dy = m_0 g \left(3 - \frac{y}{y+2}\right) dy$$

$$\begin{aligned} W &= \int_0^{h_p} m_0 g \left(3 - \frac{y}{y+2}\right) dy : \text{Note } \frac{y}{y+2} = \frac{y+2-2}{y+2} = 1 - \frac{2}{y+2} \\ &= m_0 g \int_0^{h_p} \left(3 - 1 + \frac{2}{y+2}\right) dy \\ &= m_0 g \left(2y + 2 \ln|y+2|\right) \Big|_0^{h_p} \\ &= m_0 g \left(2h_p + 2 \ln(h_p+2) - 2 \ln(2)\right) \\ &= \boxed{2m_0 g \left(h_p + \ln\left(1 + \frac{h_p}{2}\right)\right)} \end{aligned}$$



E5

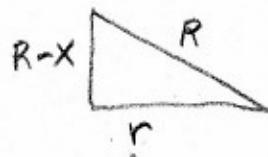
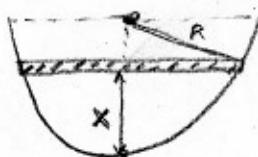


Sphere  
radius  
R

ground  
(Assume water is  
at ground level  
to begin with)

Question: How much work to half-fill the water tower?

Let  $dW$  be the work to lift the slice to  $d+x$ .  
(Let  $dV$  be the volume of that slice at  $x$ ,



$$r^2 + (R-x)^2 = R^2$$

$$r^2 = 2xR - x^2$$

$$dV = \pi r^2 dx = \pi (2xR - x^2) dx \quad (\text{volume})$$

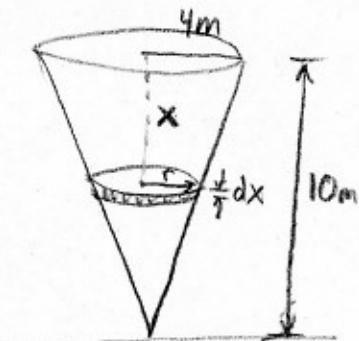
$$dm = \rho dV = \pi \rho (2xR - x^2) dx \quad (\text{mass})$$

$$dW = (g dm)(x+d)$$

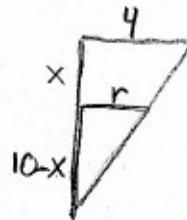
↑                      ↑  
force of gravity    distance we lifted slice.

$$\begin{aligned} W &= \int_0^R g(x+d) \pi \rho (2xR - x^2) dx \\ &= \pi \rho g \int_0^R (2x^2 R - x^3 + 2xRd - dx^2) dx \\ &= \pi \rho g \left[ \frac{2}{3} R^4 - \frac{1}{4} R^4 + R^3 d - \frac{1}{3} d R^3 \right] \\ &= \boxed{\pi \rho g \left( \frac{5}{12} R^4 + \frac{2}{3} R^3 d \right)} \end{aligned}$$

E7 (Example 4 in text) modified slightly. Find minimum work to pump water out of a full cone.



$$dm = \rho dV = \rho \pi r^2 dx$$



$$\frac{4}{10} = \frac{r}{10-x}$$

$$r = \frac{2}{5}(10-x)$$

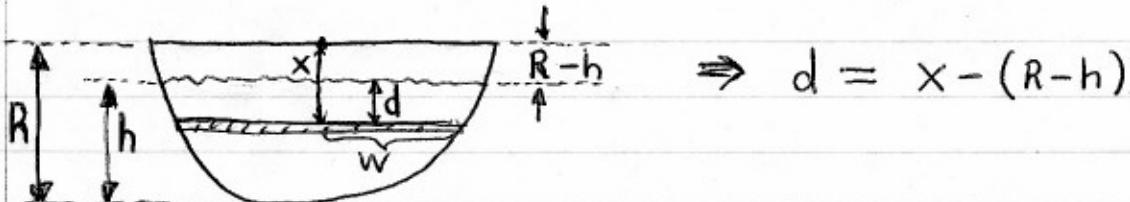
$$\begin{aligned} dW &= (dm)g \cdot x \\ &= (\rho \pi r^2 dx) g x \\ &= \rho g \pi \frac{4(10-x)^2}{25} x dx \\ &= \frac{4\pi \rho g}{25} (100x - 20x^2 + x^3) dx \end{aligned}$$

$$\begin{aligned} W &= \int_0^{10} \frac{4\pi \rho g}{25} (100x - 20x^2 + x^3) dx \\ &= \frac{4\pi \rho g}{25} \left( 50x^2 - \frac{20}{3}x^3 + \frac{1}{4}x^4 \right) \\ &= \frac{4\pi(1000)(9.8)}{25} \left( 50x^2 - \frac{20}{3}x^3 + \frac{1}{4}x^4 \right) \\ &= [4.11 \times 10^6 \text{ J}] \end{aligned}$$

Remark: this is a magic cone, we cannot just tip it over or drill a hole in the bottom. We are forced to pump out the top 😊.

(Notice the text integrates from 2 → 10 because their cone isn't assumed to be full, think about it.)

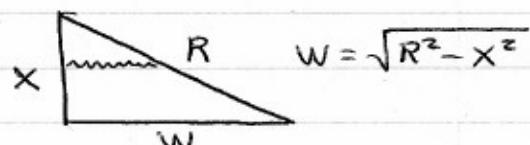
Find hydrostatic force on half-barrel pictured below, well just the end piece. The radius of barrel is  $R$  and the water of density  $\rho$  is filled to height  $h$ . Let's find force  $dF$  on a strip of area  $dA$  at position  $x$  and depth  $d$  below the surface.



From picture above we find  $P = \rho g d = \rho g (x - R + h)$ .  
(I set up the pressure wrong in notes, it is the depth that should determine the pressure, specifically  $P = \rho g d$ .) Next,

$$\text{width } 2w \quad \frac{d}{dx} dx$$

$$dA = 2w dx = 2\sqrt{R^2 - x^2} dx$$



Ok so our choice of  $x$  makes  $dA$  relatively pretty, (you can try defining  $x$  differently but it'll make the square root nasty ...) Ok, we know  $P = \frac{dF}{dA}$  so  $dF = P dA$

$$dF = \rho g (x - R + h) 2\sqrt{R^2 - x^2} dx$$

Now we just need to add-up the forces,  $R-h \leq x \leq R$

$$\begin{aligned} F &= \int_{R-h}^R 2\rho g (x - R + h) \sqrt{R^2 - x^2} dx \\ &= \int_{R-h}^R 2\rho g x \sqrt{R^2 - x^2} dx + \int_{R-h}^R 2\rho g (h - R) \sqrt{R^2 - x^2} dx \\ &= 2\rho g \underbrace{\int_{R-h}^R x \sqrt{R^2 - x^2} dx}_{u-\text{substitute}} + 2\rho g (h - R) \underbrace{\int_{R-h}^R \sqrt{R^2 - x^2} dx}_{\text{trig-substitute.}} \end{aligned}$$

HINT • The homework problem results in the same type of integration.

E8 continued

$$\begin{aligned}
 \int x \sqrt{R^2 - x^2} dx &= \int \sqrt{W} \frac{dW}{-2} \\
 &= -\frac{1}{2} \int W^{\frac{1}{2}} dW \\
 &= -\frac{1}{2} \cdot \frac{2}{3} W^{\frac{3}{2}} + C \\
 &= -\frac{1}{3} (R^2 - X^2)^{\frac{3}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 W &= R^2 - X^2 \\
 dW &= -2x dx \\
 x dx &= -\frac{dW}{2}
 \end{aligned}$$

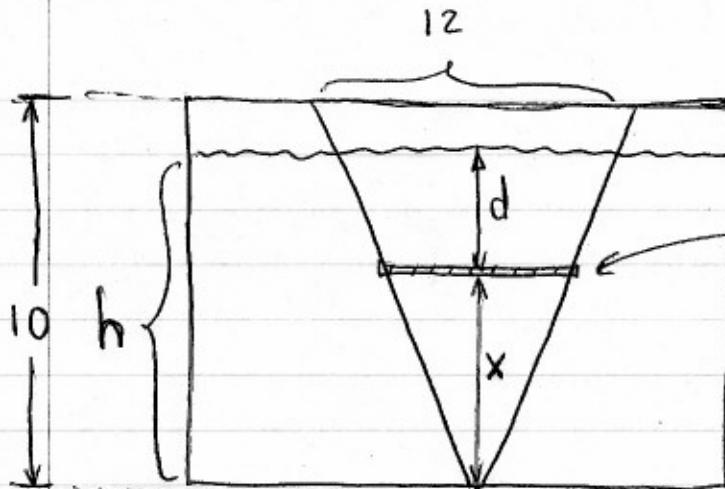
$$\begin{aligned}
 \int \sqrt{R^2 - x^2} dx &= \int (R \cos \theta) (R \cos \theta d\theta) \\
 &= \int R^2 \cos^2 \theta d\theta \\
 &= \int \frac{R^2}{2} (1 + \cos(2\theta)) d\theta \\
 &= \frac{R^2}{2} \left( \theta + \frac{1}{2} \sin(2\theta) \right) + C
 \end{aligned}$$

$$\begin{aligned}
 X &= R \sin \theta \\
 dx &= R \cos \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 R^2 - X^2 &= R^2 \cos^2 \theta \\
 \sqrt{R^2 - X^2} &= R \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 F &= 2pg \int_{R-h}^R x \sqrt{R^2 - x^2} dx + 2pg(h-R) \int_{R-h}^R \sqrt{R^2 - x^2} dx \\
 &= -\frac{2pg}{3} (R^2 - X^2)^{\frac{3}{2}} \Big|_{R-h}^R + pg(h-R) R^2 \left( \theta + \frac{1}{2} \sin(2\theta) \right) \Big|_{X=R}^{X=R-h} \\
 &= +\frac{2}{3} pg (R^2 - (R-h)^2)^{\frac{3}{2}} + pg(h-R) R^2 \left( \sin^{-1}\left(\frac{x}{R}\right) + \frac{1}{2} \sin(2 \sin^{-1}\left(\frac{x}{R}\right)) \right) \Big|_{R-h}^R \\
 &= \frac{2}{3} pg (R^2 - (R-h)^2)^{\frac{3}{2}} + pg(h-R) R^2 \left( \sin^{-1}\left(\frac{R}{R}\right) + \frac{1}{2} \sin(2 \sin^{-1}\left(\frac{R}{R}\right)) \right) \\
 &\quad - pg(h-R) R^2 \left( \sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2} \sin(2 \sin^{-1}\left(\frac{R-h}{R}\right)) \right) \\
 &= \boxed{\frac{2}{3} pg (R^2 - (R-h)^2)^{\frac{3}{2}} + pg(h-R) R^2 \left[ \frac{\pi}{2} + \frac{1}{2} \sin\left(2 \sin^{-1}\left(\frac{R-h}{R}\right)\right) \right.} \\
 &\quad \left. - \sin^{-1}\left(\frac{R-h}{R}\right) + \frac{1}{2} \sin\left(2 \sin^{-1}\left(\frac{R-h}{R}\right)\right) \right]}
 \end{aligned}$$

E9) Find the hydrostatic force on the triangular region pictured below. Assume the water of density  $\rho$  is up to level  $h$



width is  $w$  which  
clearly depends linearly  
on  $x$ .

$$w = mx + b$$

$$w(0) = m(0) + b = 0 \quad \therefore b = 0$$

$$w(10) = m(10) + 0 = 12 \quad \therefore m = \frac{12}{10} = \frac{6}{5} \quad \therefore w = \frac{6}{5}x$$

This formula for  $w$  checks because  $w(10) = \frac{6}{5}(10) = 12$  as it should.  
Then the area of strip is  $dA = wdx = \frac{6}{5}x dx$ .

Now set up the pressure  $P = \rho g d$ ,

$$x + d = h \Rightarrow d = h - x \Rightarrow P = \rho g (h - x)$$

Again the connection between force & pressure is  $F = \frac{dF}{dA}$   
so  $dF = P dA$ , thus

$$dF = \rho g (h - x) \frac{6}{5} x dx$$

Now sum the forces for the strips at  $x$  in  $0 \leq x \leq h$ ,

$$\begin{aligned} F &= \int_0^h \frac{6\rho g}{5} (hx - x^2) dx \\ &= \frac{6\rho g}{5} \left( \frac{1}{2}hx^2 - \frac{1}{3}x^3 \right) \Big|_0^h \\ &= \frac{6\rho g}{5} \left( \frac{1}{2}h^3 - \frac{1}{3}h^3 \right) \\ &= \boxed{\frac{1}{5}\rho gh^3} \end{aligned}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

### §6.5 Moments & Centers of Mass (c.o.m.)

In the discrete case we define  $\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$  the sum is over all the particles.  
 For 2 dimensional case we have

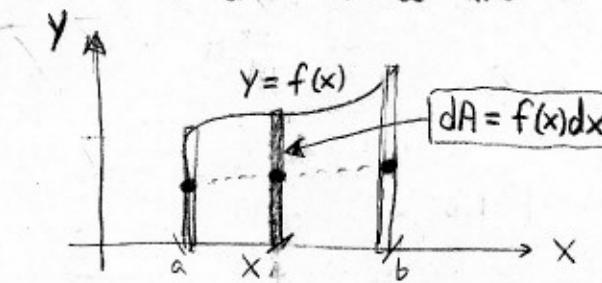
$$\vec{r}_{cm} = (x_{cm}, y_{cm})$$

$$x_{cm} = (\sum m_i x_i) / (\sum m_i) = M_x / M \quad M_x = \text{moment of inertia w.r.t } x\text{-axis.}$$

$$y_{cm} = (\sum m_i y_i) / (\sum m_i) = M_y / M \quad M = \text{total mass.}$$

What then is the generalization of this to a continuous region?

Let's see how to find the C.O.M. of the region bounded by  $y=0$ ,  $y=f(x)$  and  $x=a$  and  $x=b$ . Assume uniform density.



Each strip has its C.O.M. at  $(x, \frac{1}{2}f(x))$ . We can treat it like a bunch of particles with  $dm$  each

$$\rho = \frac{dm}{dA} \rightarrow dm = \rho dA$$

So we find the c.o.m. of this system in the natural way,

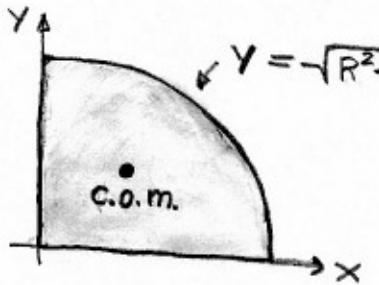
$$\bar{x} = \frac{\int_a^b \rho x f(x) dx}{\int_a^b \rho f(x) dx} \quad \text{AND} \quad \bar{y} = \frac{\int_a^b \rho \frac{1}{2}f(x) f(x) dx}{\int_a^b \rho f(x) dx}$$

Or in terms of moments & mass we have

$M_y = \rho \int x f(x) dx$ $M_x = \rho \int \frac{1}{2} [f(x)]^2 dx$ $M = \rho \int f(x) dx = \rho (\text{AREA})$	$\bar{x} = \frac{M_y}{M}$ $\bar{y} = \frac{M_x}{M}$
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We assume  $\rho$  to be a constant.

E10 Find the moments of inertia and c.o.m for a quarter-circle of uniform density  $\rho$  with radius  $R$



$$\begin{aligned} M &= \int_0^R \rho f(x) dx \\ &= \rho \int_0^R f(x) dx \\ &= \boxed{\frac{\pi}{4} \rho R^2 = M} \end{aligned}$$

mass is simply area times density.

$$\begin{aligned} M_x &= \rho \int_0^R x \sqrt{R^2 - x^2} dx \\ &= \rho \int_{R^2}^0 -\frac{1}{2} \sqrt{u} du \quad \leftarrow \begin{array}{l} u = R^2 - x^2 \\ du = -2x dx \end{array} \quad \begin{array}{l} u(R) = 0 \\ u(0) = R^2 \end{array} \\ &= \frac{\rho}{2} \cdot \frac{2}{3} u^{3/2} \Big|_0^{R^2} \\ &= \frac{\rho}{3} (R^2)^{3/2} \\ &= \boxed{\frac{1}{3} \rho R^3 = M_x} \end{aligned}$$

$$\begin{aligned} M_y &= \rho \int_0^R \frac{1}{2} (R^2 - x^2) dx \\ &= \frac{\rho}{2} \left( R^2 x - \frac{1}{3} x^3 \right) \Big|_0^R \\ &= \boxed{\frac{1}{3} \rho R^3 = M_y} \end{aligned}$$

Now we can find the center of mass,

$$x_{cm} = \frac{M_y}{M} = \frac{\frac{1}{3} \rho R^3}{\frac{\pi}{4} \rho R^2} = \frac{4}{3\pi} R \approx 0.4244 R$$

$$y_{cm} = \frac{M_x}{M} = \frac{\frac{1}{3} \rho R^3}{\frac{\pi}{4} \rho R^2} = \frac{4}{3\pi} R \approx 0.4244 R$$

From the symmetry of the region we could have anticipated  $x_{cm} = y_{cm}$ .