

Certain 1<sup>st</sup> order ode's can be seen to have the form

$$\frac{dy}{dx} = g(x) f(y)$$

In which case we can separate the variables

$$\frac{dy}{f(y)} = g(x) dx$$

And then integrate

$$\int \frac{dy}{f(y)} = \int g(x) dx$$

When you can actually do these integrals this will implicitly (and sometimes explicitly) define  $y$  in terms of  $x$ ; that means you found a sol<sup>n</sup>!

Pf/ 
$$\int \frac{dy}{f(y)} = \int \frac{g(x) f(y) dx}{f(y)}$$

$$= \int g(x) dx //$$

: Changing variables to  $x$   
 $dy = g(x) f(y) dx$  by assumption

**E1**  $\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k dt$  then integrate

$$\int \frac{dy}{y} = \int k dt \Rightarrow \ln y = kt + c \quad (\text{implicit sol}^n)$$

Thus  $y(t) = e^{kt+c} = e^c e^{kt} = \boxed{y_0 e^{kt} = y(t)}$  (explicit sol<sup>n</sup>)

If  $y(0) = 3$  find the sol<sup>n</sup>

$$y(0) = y_0 e^{k(0)} = \boxed{y_0 = 3} \Rightarrow \underline{y(t) = 3e^{kt}}$$

(This is why  $y_0$  is good notation here)

Remark:  $k > 0$  exponential growth  $ey^t$ . (More on this later)  
 $k < 0$  exponential decay  $ey^{-t}$

**E2**  $\frac{dy}{dx} = a^{x+y}$  : find sol<sup>n</sup> thru sep. of variables

$\int a^{-y} dy = \int a^x dx$  : separate then integrate.

$\frac{-1}{\ln(a)} a^{-y} = \frac{1}{\ln(a)} a^x + \tilde{c}$

$a^{-y} = -a^x - \ln(a)\tilde{c} = c - a^x$  : want to solve for  $y$ .

$\ln(a^{-y}) = \ln(c - a^x)$

: it is crucial to insure we are taking the  $\ln$  of positive quantities!  
that is why we moved the minus sign to the other side.

$-y \ln(a) = \ln(c - a^x)$

$y = \frac{-\ln(c - a^x)}{\ln(a)}$

Notice that  $c$  is arbitrary and can only be specified if we supply further demands (an initial or boundary condition)

**E3**  $\frac{du}{d\theta} = \frac{2\theta + \sec^2\theta}{2u}$  find sol<sup>n</sup> with  $u(0) = -5$

$2u du = (2\theta + \sec^2\theta) d\theta$  : separated variables, now integrate,

$u^2 = \theta^2 + \tan\theta + C$  : an implicit sol<sup>n</sup>.

$\therefore u = \pm \sqrt{\theta^2 + \tan\theta + C}$  : an explicit sol<sup>n</sup>.

$u(0) = \pm \sqrt{C} = -5$

$\Rightarrow -\sqrt{C} = -5$  (Must choose negative sqrt. sol<sup>n</sup>)

$\Rightarrow C = 25$

$u(\theta) = -\sqrt{\theta^2 + \tan\theta + 25}$

**E4** Find implicit sol<sup>n</sup> to  $\frac{dy}{dx} = \frac{\cos(x)}{y^4 + y^2 + 23}$ . Also find Equilibrium sol<sup>n</sup>'s.

$$\int (y^4 + y^2 + 23) dy = \int \cos(x) dx$$

$$\boxed{\frac{1}{5} y^5 + \frac{1}{3} y^3 + 23y = \sin(x) + C} \leftarrow \text{this implicitly solves the DE.}$$

Here we cannot simply solve for  $y$  as a fct. of  $x$ . The sol<sup>n</sup> is still useful because it describes how  $x$  &  $y$  are related along the sol<sup>n</sup> curves. Next find Equilibrium Sol<sup>n</sup>'s

$$\frac{dy}{dx} = 0 \Rightarrow \frac{\cos(x)}{y^4 + y^2 + 23} = 0$$

$$\Rightarrow \cos(x) = 0$$

$$\Rightarrow \boxed{x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}}$$

**E5**  $mg = ma = m \frac{dv}{dt} = m \frac{dx}{dt} \frac{dv}{dx} = mv \frac{dv}{dx}$   
 Lets solve  $mg = mv \frac{dv}{dx}$  to find velocity as function of  $x$

Cancel  $m$  to begin

$$g = v \frac{dv}{dx}$$

$$\int g dx = \int v dv$$

$$gx = \frac{1}{2} v^2 + C$$

Now if  $v(x_0) = v_0$  then  $gx_0 = \frac{1}{2} v_0^2 + C \therefore C = gx_0 - \frac{1}{2} v_0^2$

Yielding that,

$$gx = \frac{1}{2} v^2 + gx_0 - \frac{1}{2} v_0^2$$

$$g(x - x_0) = \frac{1}{2} (v^2 - v_0^2) \Rightarrow$$

$$\boxed{v^2 = v_0^2 + 2g(x - x_0)}$$

relates velocity & position w/o reference to time.

**E6** The falling Raindrop: Imagine a drop falling thru a cloud gathers water as it falls, let  $m(t)$  be its varying mass. Further assume as the drop gets bigger it gathers more & more mass proportionate to it's mass;  $\frac{dm}{dt} = km$  for  $k > 0$ .

$F = ma$  is more generally  $F = \frac{dp}{dt}$  when  $m$  varies.

$$F = \frac{dp}{dt} = \frac{d}{dt}(mv) = mg \quad (\text{fall's due to gravity})$$

$$\left(\frac{dm}{dt}\right)v + m\frac{dv}{dt} = mg$$

$$kmv + m\frac{dv}{dt} = mg$$

$$\frac{dv}{dt} = \frac{mg - kmv}{m} = g - kv$$

$$\therefore \frac{dv}{kv - g} = -dt \Rightarrow \frac{dv}{v - g/k} = -kdt$$

Integrate both sides,  $\ln|v - g/k| = -kt + \tilde{c}$  and exponentiate,

$$v - g/k = e^{-kt + \tilde{c}} = ce^{-kt}$$

$$v(t) = ce^{-kt} + g/k$$

The terminal velocity would be

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} (ce^{-kt} - g/k) = \boxed{g/k}$$

Remark: We took down as positive direction & Assumed no friction besides the water growth. Physically this amounts to under friction!

$$m\frac{dv}{dt} = ma = mg - kmv$$

When  $v = g/k$  we have  $mg - km\left(\frac{g}{k}\right) = 0$

terminal velocity happens when the forces balance

• Orthogonal Trajectories (O.T.)

Given a family of curves an O.T. is a curve which is orthogonal to each member of the family; meaning at the points of interception the tangents are orthogonal (aka perpendicular, if  $m$ -slope then  $-\frac{1}{m}$  is  $\perp$  line's slope)

**E7**  $x^2 + y^2 = R^2$  defines a circle for each  $R > 0$ . Let's find the orthogonal trajectories to this family of curves, diff. implicitly

$$2x + 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x}{y} \quad (\text{for circles})$$

Now then the O.T must have  $\frac{dy}{dx} = \frac{-1}{-x/y} = \frac{y}{x}$  hence

$$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + C$$
$$\Rightarrow y = e^{\ln(x)+C} = e^{\ln(x)} e^C = x e^C$$

Thus as pictured in figure 8 on pg. 516  $y = mx$  are truly the OT's to circles.

**E8**  $x^2 - y^2 = k \Rightarrow 2x - 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$

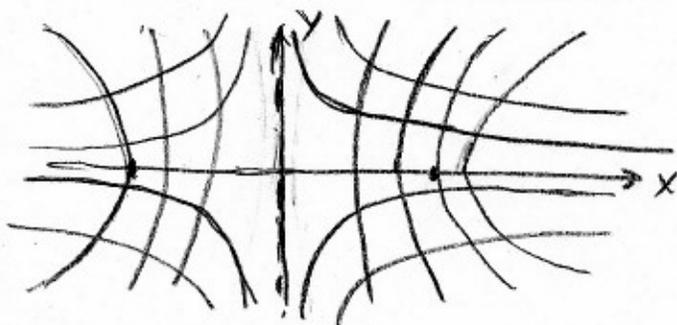
Hence the O.T. will have  $\frac{dy}{dx} = \frac{-1}{x/y} = -\frac{y}{x}$

$$\therefore \frac{dy}{-y} = \frac{dx}{x} \Rightarrow -\ln|y| = \ln|x| + \tilde{C}$$

$$\Rightarrow \ln\left(\frac{1}{|y|}\right) = \ln(\tilde{C}x)$$

$$\Rightarrow \frac{1}{|y|} = \tilde{C}|x|$$

$$\Rightarrow \boxed{y = \pm \frac{C}{x}} \text{ is the O.T.}$$



$x^2 - y^2 = k$  is a hyperbola

$$y=0 \quad x = \pm k$$

$$x=0 \quad y \text{ d.n.e}$$

## E9 Mixing Problems

176

Consider some tank of fixed volume with some substance entering/exiting the tank, let  $Y(t)$  be the amount of the substance at time  $t$  in the tank.

(I'll work #35 for you)

Pure Water  $\rightarrow$  10 L/min



15 kg of salt at  $t=0$

Let  $Y(t)$  = kg of salt in tank at time  $t$

$$\frac{dY}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 0 - \left(10 \frac{\text{L}}{\text{min}}\right) \cdot \left(\frac{Y(t)}{1000\text{L}}\right)$$

$$= -\frac{1}{100} Y(t) / \text{min} \leftarrow \text{kg/min makes sense}$$

$$= -Y/100$$

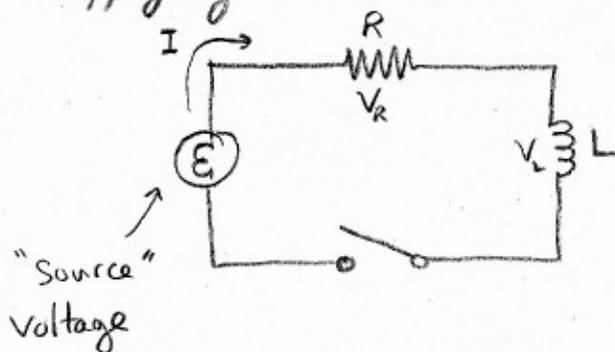
Thus  $\frac{dY}{Y} = -\frac{dt}{100} \quad \therefore \ln(Y) = -\frac{t}{100} + \tilde{c} \quad \therefore Y(t) = Y_0 e^{-t/100}$

$Y(0) = 15\text{kg} = Y_0 \quad \therefore Y(t) = 15e^{-t/100} \text{ kg}$

So after 20 minutes  $Y(20) = 15e^{-20/100} = 12.28 \text{ kg at } t=20 \text{ min}$

## E10 The RL-Circuit

Applying Kirchhoff's Rules & a def<sup>n</sup> or two we find



$$\mathcal{E} = V_R + V_L \quad (\text{Kirchhoff})$$

$$V_L = L \frac{dI}{dt} \quad (\text{Resists change in current})$$

$$V_R = IR \quad (\text{Ohm's Law})$$

$$\mathcal{E} = IR + L \frac{dI}{dt}$$

Now if  $\mathcal{E} = \text{constant}$  and  $I(0) = 0$  find  $I(t)$ ,

$$\frac{\mathcal{E} - IR}{L} = \frac{dI}{dt} \quad \therefore \int \frac{dI}{\mathcal{E} - IR} = \int \frac{dt}{L}$$

$$\int \frac{dI}{\mathcal{E} - IR} = -\frac{1}{R} \ln(\mathcal{E} - IR)$$

$$\int \frac{dt}{L} = \frac{t}{L} + C$$

$$\text{Thus } -\frac{1}{R} \ln(\mathcal{E} - IR) = \frac{t}{L} + \bar{C} \Rightarrow \mathcal{E} - IR = \bar{C} e^{-\frac{R}{L}t} \text{ that is}$$

$$I = \frac{\mathcal{E}}{R} (1 + C e^{-\frac{R}{L}t})$$

$$I(0) = \frac{\mathcal{E}}{R} (1 + C) = 0 \quad \therefore C = -1$$

$$\therefore \boxed{I(t) = \frac{\mathcal{E}}{R} (1 - e^{-\frac{R}{L}t})}$$

Remark: the limiting current as  $t \rightarrow \infty$  is  $\mathcal{E}/R$ . That is physically speaking the inductor is a short circuit for "long" times ( $\tau = L/R$  then  $5\tau \approx \infty$  pragmatically speaking.)