

This is why we talk about series at all. In previous sections we have employed some rather indirect reasoning to connect a function & its corresponding series representation.

Theorem (Taylor) If  $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$  for  $|x-a| < R$  then the coefficients  $c_n$  are given by  $c_n = \frac{f^{(n)}(a)}{n!}$ . Moreover

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

That is, noting  $f^{(0)}(x) \equiv f(x)$ ,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots$$

Proof: We'll simply use the term-by-term diff. Thm multiple times.

Assume that  $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$  for  $|x-a| < R$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots \Rightarrow f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots \Rightarrow f'(a) = c_1$$

$$f''(x) = 2c_2 + 6c_3(x-a) + \dots \Rightarrow f''(a) = 2c_2$$

$$f'''(x) = 6c_3 + \dots \Rightarrow f'''(a) = 6c_3$$

⋮

$$\begin{aligned} f^{(n)}(x) &= \left(\frac{d}{dx}\right)^n \left[ \sum_{k=0}^{\infty} c_k (x-a)^k \right] \\ &= \sum_{k=n}^{\infty} c_k \cdot k(k-1)(k-2)\cdots(k-(n-1)) (x-a)^{k-n} \end{aligned}$$

$$\text{Then } f^{(n)}(a) = c_n \cdot n(n-1)(n-2)\cdots(2)(1)(a-a)^0 \Rightarrow f^{(n)}(a) = c_n \cdot n!$$

Therefore  $c_n = \frac{f^{(n)}(a)}{n!}$  as claimed and substituting back into  $f(x)$  we find

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(Assuming  $f(x)$  has a power series expansion!)

Remark: WHEN  $a=0$  the TAYLOR SERIES EXPANSION BECOMES

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In which case we can call it the MACLAURIN SERIES

Discussion: For any differentiable func. it's fairly clear that we can construct a taylor series for that func, we simply differentiate and evaluate then construct  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ .

In fact by the last Th<sup>m</sup> we know that if  $f(x)$  has a power series expansion then it matches the TAYLOR EXPANSION. A question we should answer then is given a differentiable function when does it have a TAYLOR SERIES EXPANSION? DOES IT ALWAYS HAVE ONE? WELL NO, see next example after the th<sup>n</sup>.

Th<sup>n</sup> (TAYLOR'S TH<sup>n</sup>) If  $f$  is differentiable at least  $n+1$  times on an open interval  $I$  containing  $a$  then for each  $x \in I$   $\exists c$  between  $x$  and  $a$  such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

When  $R_n(x) \rightarrow 0 \quad \forall x \in I$  as  $n \rightarrow \infty$  we say the Taylor Series generated by  $f$  converges to  $f$  on  $I$ , in which case

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

E)  $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$  differentiable  $\not\Rightarrow$  analytic

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

this function is differentiable at zero but its not equal to its taylor series expanded about zero.

$$f(x) = f(0) + f'(0)x + \dots = 0 + 0 \cdot x = 0 \quad \text{yet } f(x) \neq 0 !$$

Remark: Taylor's Th<sup>n</sup> is a generalization of the mean-value th<sup>n</sup>, and much like that th<sup>n</sup> it only tells us  $\exists$  some  $c$  in  $I$ , but how to find  $c$  such that  $R_n(x) = \frac{f^{(n+1)}}{(n+1)!} (x-a)^{n+1}$ ? Well please tell me if you know! What we can say is

Th<sup>n</sup> (TAYLOR'S INEQUALITY). If  $|f^{(n+1)}(x)| \leq M$  for  $|x-a| < R$  then the remainder  $R_n(x)$  of the TAYLOR SERIES is bounded by

$$0 \leq |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ for } |x-a| < R$$

Pf: (See Text).

Remark: With Taylor's Inequality we no longer need to calculate  $R_n(x)$  explicitly! We merely need to find an  $M$ .

Then if we can show  $\frac{M}{(n+1)!} |x-a|^{n+1} \rightarrow 0$  we can apply squeeze th<sup>n</sup> to conclude  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ .

- Let's find the MacLaurin series rep. of  $e^x$  (Assume  $R_n \rightarrow 0$  as  $n \rightarrow \infty$ )

**E4** Let  $f(x) = e^x$  then  $f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x$ . Thus the taylor series about zero for  $e^x$  is simple,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{e^0}{n!} x^n \\ &= \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = e^x \end{aligned}$$

← MacLaurin series for the exponential func.

Remark: to be more rigorous we ought to prove that the Taylor series for  $e^x$  does indeed converge to  $e^x$ . This amounts to showing  $R_n \rightarrow 0$ . (We'll avoid this for now)

**E3** Prove  $f(x) = \sin(x)$  is rep. by it's Maclaurin series  $\forall x$ ,

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f^{(IV)}(x) = \sin(x)$$

Thus  $f^{(n)}(x) = \pm \sin(x)$  or  $\pm \cos(x)$   $\therefore |f^n(x)| \leq 1 = M$ .

By TAYLOR'S INEQ, Th<sup>n</sup>,

$$0 \leq |R_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus by Squeeze th<sup>n</sup>  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  (for all  $x$ ).

Moreover, the Maclaurin series is easily calculated.

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sin(0) + \cos(0) \cdot x - \frac{\sin(0)}{2!} x^2 - \frac{\cos(0)}{3!} x^3 + \dots \\ &= \boxed{x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin(x)} \end{aligned}$$

**E4** Ok, enough about existence, let's assume they exist, not an unreasonable step for most fncts. we can think of. Let  $f(x) = \cos(x)$

$$f(x) = \cos(x)$$

$$f(0) = 1$$

$$f'(x) = -\sin(x)$$

$$f'(0) = 0$$

$$f''(x) = -\cos(x)$$

$$f''(0) = -1$$

$$f'''(x) = \sin(x)$$

$$f'''(0) = 0$$

$$f^{(IV)}(x) = \cos(x)$$

$$f^{(IV)}(0) = 1$$

$$\text{Hence } \cos(x) = 1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = \cos(x)}$$

**E5** Geometric Series: Given  $f(x) = \frac{1}{1-x}$  what do we find,

223

$$f(x) = \frac{1}{1-x}$$

$$f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f'(0) = 1$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f''(0) = 2!$$

$$f'''(x) = \frac{-3+2+1}{(1-x)^4}$$

$$f'''(0) = 3!$$

⋮

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

$$f^{(n)}(0) = n!$$

Checking consistency.  
we already knew the result here w/o work

$$\text{Hence } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

$$\text{That is } \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\boxed{\text{E6} \quad \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x})}$$

Let us find Taylor expansion of  $\cosh(x)$  about  $x=0$ ,

$$f(x) = \cosh(x) \quad f(0) = \frac{1}{2}(e^0 + e^0) = 1$$

$$f'(x) = \sinh(x) \quad f'(0) = \frac{1}{2}(e^0 - e^0) = 0$$

$$f''(x) = \cosh(x) \quad f''(0) = 1$$

$$f'''(x) = \sinh(x) \quad f'''(0) = 0$$

$$f''''(x) = \cosh(x) \quad f''''(0) = 1$$

$$\text{Hence } \boxed{\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

$$\text{Very similarly } \boxed{\sinh(x) = 1 + x + \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$$

Remark: just like  $\cos(x)$  and  $\sin(x)$  just no alternating signs.

E7  $f(x) = \sin^2(x)$ . One way is  $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$

(224)

Using E4 with  $2x$  in place of  $x$  we find

$$\cos(2x) = 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6$$

Thus subst. this into identity above gives

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}(1 - [1 - 2x^2 + \frac{16}{24}x^4 - \frac{64}{720}x^6 + \dots]) \\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots\end{aligned}$$

A second method is to multiply the series for  $\sin(x)$

$$\begin{aligned}\sin^2(x) &= (x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots)(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots) \\ &= x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \dots - \frac{1}{3!}x^4 + \frac{1}{(3!)^2}x^6 + \dots + \frac{1}{5!}x^6 + \dots \\ &= x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right)x^6 + \dots \\ &= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots\end{aligned}$$

A third method is to simply Taylor expand,

$$f(x) = \sin^2(x) \quad f(0) = 0$$

$$f'(x) = 2\sin(x)\cos(x) \quad f'(0) = 0$$

$$f''(x) = 2(\cos^2(x) - \sin^2(x)) \quad f''(0) = 2$$

$$f'''(x) = -4\sin(x)\cos(x) - 4\sin(x)\cos(x) \quad f'''(0) = 0$$

$$f''''(x) = -8(\cos^2(x) - \sin^2(x)) \quad f''''(0) = -8$$

$$\text{Thus } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{2x^2}{2!} - \frac{8x^4}{4!} + \dots = x^2 - \frac{1}{3}x^4 - \dots$$

Which method do you think is best?

Obviously it depends on the example and what we're asked to find.

E8 Expand  $f(x) = \sqrt{x}$  around  $a = 4$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}}$$

$$f''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = -\left(\frac{1}{2}\right)^2 \frac{1}{(\sqrt{x})^2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} = 3\left(\frac{1}{2}\right)^3 \frac{1}{(\sqrt{x})^5}$$

$$f''''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-\frac{7}{2}} = -3 \cdot 5 \left(\frac{1}{2}\right)^4 \frac{1}{(\sqrt{x})^7} \quad f''''(4) = \frac{-15}{2^8}$$

$$f(4) = 2$$

$$f'(4) = \frac{1}{4}$$

$$f''(4) = -\frac{1}{32}$$

$$f'''(4) = \frac{3}{28}$$

Thus we find

$$f(x) = \sum \frac{f^{(n)}(4)}{n!} (x-4)^n$$

$$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4 + \dots = \sqrt{x}}$$

E9 Find Taylor exp. of  $f(x) = x^3 + 3x^2 + 3x + 1$  about  $x = 0$   
and  $x = -1$ . Notice  $f(0) = 1$  and  $f(-1) = 0$  and

$$f'(x) = 3x^2 + 6x + 3 \quad f'(0) = 3 \quad f'(-1) = 0$$

$$f''(x) = 6x + 6 \quad f''(0) = 6 \quad f''(-1) = 0$$

$$f'''(x) = 6 \quad f'''(0) = 6 \quad f'''(-1) = 6$$

$$f''''(x) = 0$$

about zero  $\rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + 3x + \frac{6}{2!}x^2 + \frac{6}{3!}x^3 = x^3 + 3x^2 + 3x + 1$

about  $x = -1$   $\rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} (x+1)^n$

$$= 0 + 0 \cdot (x+1) + \frac{0}{2!} (x+1)^2 + \frac{6}{3!} (x+1)^3$$

$$= \boxed{(x+1)^3 = x^3 + 3x^2 + 3x + 1 = f(x)}$$

Remark: the best  $\infty$ -polynomial approximation to a polynomial is itself. Surprise, surprise.  
Power series are basically polynomials of infinite order.

**E10** Find MacLaurin series for  $\tan(x) = f(x)$ . Find 1<sup>st</sup> 2 non-zero terms.

$$\begin{array}{ll} f(x) = \tan(x) & f(0) = 0 \\ f'(x) = \sec^2(x) & f'(0) = 1 \\ f''(x) = 2\sec^2(x)\tan(x) & f''(0) = 0 \end{array}$$

$$f'''(x) = 4\sec^2(x)\tan^2(x) + 2\sec^4(x) \quad f'''(0) = 2$$

Hence  $\tan(x) = x + \frac{2}{3!}x^3 + \dots = \boxed{x + \frac{1}{3}x^3 + \dots = \tan(x)}$

We could differentiate further if we want to generate higher order terms.

Summary: The following MacLaurin series you should know

	I.O.C
$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + u^3 + \dots$	(-1, 1)
$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3$	(-\infty, \infty)
$\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \dots$	(-\infty, \infty)
$\cos(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 - \dots$	(-\infty, \infty)

Remark: for many problems you can use these basic series rather than explicitly generating the Taylor exp.

**E11**  $x \sin(\frac{x}{2}) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{[\frac{1}{2}x]^{2n+1}}{(2n+1)!}$

"sigma notation"

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \frac{x^{2n+2}}{(2n+1)!}} = x \left( \frac{x}{2} - \frac{1}{3!} \left(\frac{x}{2}\right)^3 + \frac{1}{5!} \left(\frac{x}{2}\right)^5 - \dots \right)$$

$$= \boxed{\frac{1}{2}x^2 - \frac{1}{48}x^4 + \frac{1}{3840}x^6 - \dots}$$

1<sup>st</sup> three non-zero terms.