MA 241-003 - TEST I - INTEGRATION

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Please clearly show all your work. Box or circle answers. Thanks.

$\boxed{1.\ 10\text{pts}}$ Given that $f(x) = \begin{cases} a & 1 < x \\ ax & x \geq 1 \end{cases}$, where a is some arbitrairy but unknown constant, show that $\int_0^2 f(x)\,dx = \frac{5a}{2}$.

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 a\,dx + \int_1^2 ax\,dx$$

$$= ax\Big|_0^1 + \tfrac{1}{2}ax^2\Big|_1^2$$

$$= a(1-0) + \tfrac{1}{2}a(2^2 - 1^2)$$

$$= a + \tfrac{3}{2}a$$

$$= \boxed{\frac{5a}{2}}$$

this is like #5 on test 1 review

$\boxed{2.\ 10\text{pts.}}$ Calculate $\frac{d}{dx}\left(\int_{\sin(x)}^{x} g(u)\,du\right)$. Assume g has an antiderivative.

$$\frac{d}{dx}\left[\int_{\sin(x)}^{x} g(u)\,du\right] = \frac{d}{dx}\Big[G(x) - G(\sin(x))\Big]$$

: assuming that $G'(x) = g(x)$, its the antiderivative.

$$= G'(x) - G'(\sin(x)) \cdot \cos(x)$$

: chain-rule

$$= \boxed{g(x) - g(\sin(x)) \cdot \cos(x)}$$

same as $\boxed{E1} \rightarrow \boxed{E5}$ on 96 $\rightarrow$ 97 especially E5

3. 20pts. Use u-substitution to solve the following integrals,

a.) $\int x^3(3+x^4)^6\,dx = \int x^3 u^6 \frac{du}{4x^3}$

$u = 3 + x^4$
$du = 4x^3 dx$
$\Rightarrow dx = \frac{du}{4x^3}$

$= \frac{1}{4}\int u^6\,du$

$= \frac{1}{4}\left(\frac{1}{7}u^7\right) + C$

$= \boxed{\frac{1}{28}(3+x^4)^7 + C}$

almost # 3 of §5.5 webassign

b.) $\int \frac{\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{-du}{1+u^2}$

$u = \cos(x)$
$du = -\sin(x)\,dx$
$-du = \sin(x)\,dx$

$= -\tan^{-1}(u) + C$

$= \boxed{-\tan^{-1}(\cos(x)) + C}$

this is #11 of webassign 5.5

3 continued (U- substitution)

c.) $\int_2^3 \frac{1}{x-4}\,dx = \int_{-2}^{-1} \frac{1}{u}\,du$

$u = x-4$
$u(3) = 3-4 = -1$
$u(2) = 2-4 = -2$
$du = dx$

$= \ln|u| \Big|_{-2}^{-1}$

$= \ln|-1| - \ln|-2|$

$= \ln(1) - \ln(2)$

$= \boxed{-\ln(2)}$

#10 on review sheet & I warned most of you that your notation was incorrect on the turn-in work.

d.) $\int \sec(\theta)\,d\theta = \int \frac{du}{u}$

$u = \sec\theta + \tan\theta$
$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$
$= \sec\theta(\tan\theta + \sec\theta)\,d\theta$
$= (\sec\theta)\,u\,d\theta$
$\Rightarrow \frac{du}{u} = \sec\theta\,d\theta$

$= \ln|u| + C$

$= \boxed{\ln|\sec\theta + \tan\theta| + C}$

Example 20 on 104. Also this came up numerous other places in notes

$\boxed{4.\ 15\text{pts}}$ Trigonometric Substitution, leave answer as algebraic expression.

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta \cdot \cancel{3\cos\theta}\,d\theta}{\cancel{3\cos\theta}}$$

$$\boxed{\begin{aligned} x &= 3\sin\theta \\ \sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} \\ &= \sqrt{9(1-\sin^2\theta)} \\ &= 3\sqrt{\cos^2\theta} \\ &= 3\cos\theta \\ dx &= 3\cos\theta\,d\theta \end{aligned}}$$

$$= 27\int \sin^2\theta \sin\theta\,d\theta$$

$$= 27\int (1-\cos^2\theta)\sin\theta\,d\theta$$

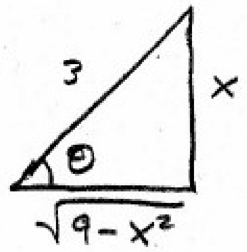
$$= 27\int (1-u^2)(-du) \longleftarrow \boxed{\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\,d\theta \\ -du &= \sin\theta\,d\theta \end{aligned}}$$

$$= 27\int (u^2-1)\,du$$

$$= 27\left(\tfrac{1}{3}u^3 - u\right) + C$$

$$= 9\cos^3\theta - 27\cos\theta + C$$

$$= 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - 27\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$



$$= \boxed{\tfrac{1}{3}(9-x^2)^{3/2} - 9\sqrt{9-x^2} + C}$$

this is exactly example 4 from notes, although my sol° while equivalent uses $x = 3\sin\theta$ instead of $3\cos\theta = x$

5. 15pts. Integration by Parts

$$\int \underbrace{\sin^{-1}(x)}_{u}\, \underbrace{dx}_{dv} = x\sin^{-1}(x) - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

I.B.P

$u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \dfrac{dx}{\sqrt{1-x^2}}$ $\quad v = x$

$$= x\sin^{-1}(x) - \int \frac{-dw/2}{\sqrt{w}}$$

$w = 1 - x^2$

$dw = -2x\,dx$

$\Rightarrow x\,dx = -\dfrac{dw}{2}$

$$= x\sin^{-1}(x) + \frac{1}{2}\int w^{-1/2}\,dw$$

$$= x\sin^{-1}(x) + \frac{1}{2}\,\frac{w^{1/2}}{1/2} + C$$

$$= \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

this was #3 of the 5.6 webassign

6. 20pts. PARTIAL FRACTIONS (15pts for a.) & 5pts. for b.))

a.) $\displaystyle\int \frac{4x^2-1}{x^2(x-1)}\,dx$

part a. similar to problems in lecture & webassign.

We decompose using partial fractions,

$$\frac{4x^2-1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$4x^2-1 = Ax(x-1) + B(x-1) + Cx^2$$

$x=0$ | $-1 = -B \quad \therefore \boxed{B=1}$

$x=1$ | $3 = C \quad \therefore \boxed{C=3}$

$x=2$ | $15 = 2A + B + 4C \Rightarrow 15 = 2A + 1 + 12$

$\Rightarrow 2 = 2A \quad \therefore \boxed{A=1}$

Therefore,

$$\int \frac{4x^2-1}{x^2(x-1)}\,dx = \int \frac{dx}{x} + \int \frac{dx}{x^2} + 3\int \frac{dx}{x-1}$$

$$= \ln|x| - \frac{1}{x} + 3\int \frac{du}{u} \qquad \boxed{\begin{aligned} u &= x-1 \\ du &= dx \end{aligned}}$$

$$= \ln|x| - \frac{1}{x} + 3\ln|u| + C$$

$$= \boxed{\ln|x| - \frac{1}{x} + 3\ln|x-1| + C}$$

part b similar to E4 on 122

b.) Give partial fractal decomposition of $f(x)$ below. Do NOT DETERMINE the coefficients $A, B, \ldots$.

$$f(x) = \frac{(x^2+2)(x-5)}{x^3(x^2+1)(x+3)(x^2+x+2)^2}$$

$$= \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{F}{x+3} + \frac{Gx+H}{x^2+x+2} + \frac{Ix+J}{(x^2+x+2)^2}}$$

Where we noticed x^2+1 and x^2+x+2 are irreducible quadratics, since $b^2-4ac = 1-4$ and $1-8$ respectively & both discriminants were negative.