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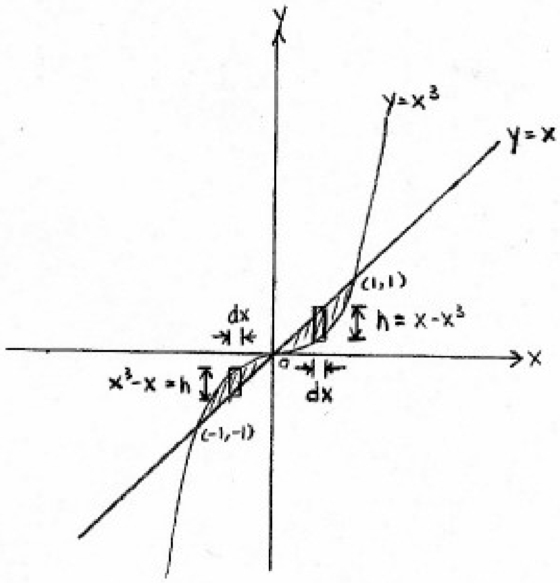
MA241-006: Calculus II

Instructor: Mr. James Cook
Test: #2 Form A
Date: Monday, Febuary 27, 2006

Directions: You **must** show **ALL** your work to receive credit.

1. (25 pts) Find the area of the region bounded by $y = x$ and $y = x^3$. For full credit your solution should include a graph and diagrams of the appropriate infinitesimal rectangle(s).



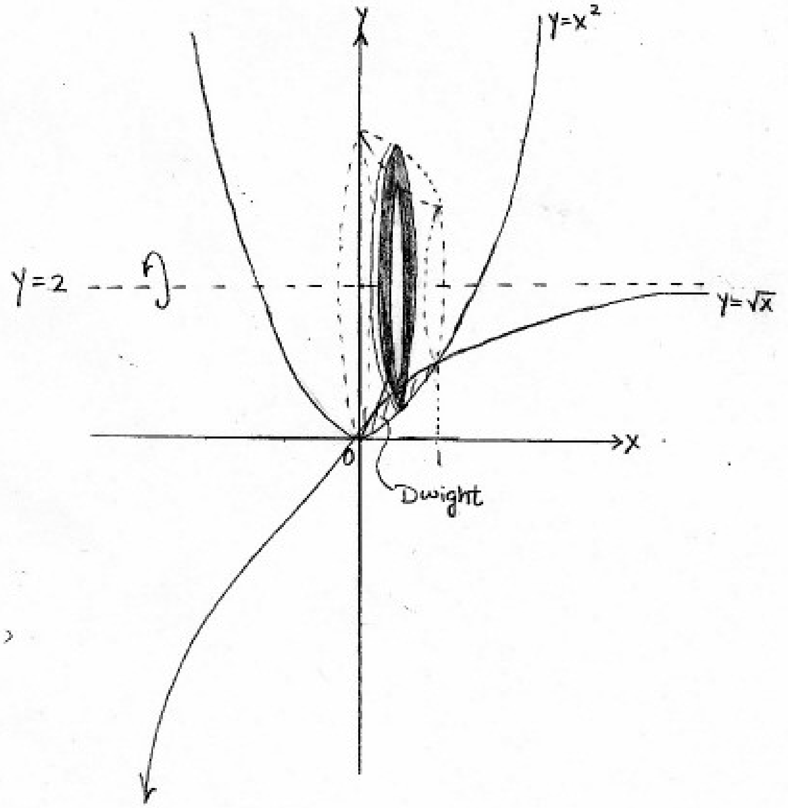
$$\text{Area} = \int_{-1}^{0} x^3 - x\,dx + \int_0^1 x - x^3 dx$$

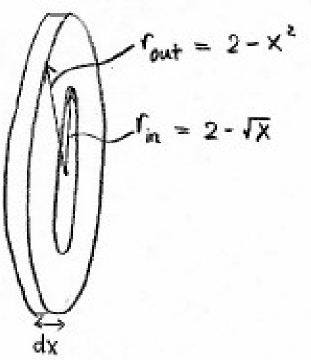
$$= 2\int_0^1 x - x^3 dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = 2\left(\frac{1}{4}\right) = \frac{1}{2}.$$

2. (25pts) Let us give the region bounded by $y = x^2$ and $y = \sqrt{x}$ the name *Dwight*. Find the volume of the solid obtained from rotating *Dwight* around $y = 2$. For full credit your solution should include a graph of the region as well as some picture of the typical approximating washer with r_{in} and r_{out} as they relate to the region in question. In other words, present your solution roughly as I have in class, indicate where the final integral came from. The set-up is worth most of the points this problem.





$$\text{Volume} = \pi \int_0^1 r_{out}^2 - r_{in}^2 \, dx$$

$$= \pi \int_0^1 (2-x^2)^2 - (2-\sqrt{x})^2 \, dx$$

$$= \pi \int_0^1 4 - 4x^2 + x^4 - (4 - 4\sqrt{x} + x) \, dx$$

$$= \pi \int_0^1 x^4 - 4x^2 - x + 4\sqrt{x} \, dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{4x^3}{3} - \frac{x^2}{2} + \frac{8}{3} x^{3/2} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{5} - \frac{4}{3} - \frac{1}{2} + \frac{8}{3} \right)$$

$$= \pi \, \frac{6 - 40 - 15 + 80}{30}$$

$$= \frac{31}{30} \pi$$

3. (25 pts) Consider the curve in the xy-plane described parametrically by the equations:

$$x = e^{\theta}(\sin\theta - \cos\theta)$$
$$y = e^{\theta}(\sin\theta + \cos\theta)$$
$$0 \leq \theta \leq \ln 2$$

Find the arclength of this curve. (If you forgot the formula for arclength you can buy it for 3pts during the test)

$$\frac{dx}{d\theta} = e^{\theta}(\sin\theta - \cos\theta) + e^{\theta}(\cos\theta + \sin\theta)$$ (product rule)

$$= 2e^{\theta}\sin\theta$$

$$\frac{dy}{d\theta} = e^{\theta}(\sin\theta + \cos\theta) + e^{\theta}(\cos\theta - \sin\theta)$$

$$= 2e^{\theta}\cos\theta$$

$$\text{Arclength} = \int_0^{\ln 2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\ln 2} \sqrt{4e^{2\theta}\sin^2\theta + 4e^{2\theta}\cos^2\theta}\, d\theta$$

$$= \int_0^{\ln 2} 2e^{\theta}\, d\theta$$

$$= 2e^{\theta}\Big|_0^{\ln 2}$$

$$= 2\left(e^{\ln 2} - e^{0}\right)$$

$$= 2\,(2-1) = 2.$$

4. (25 pts) Let $f(x) = kx^2(1-x)$ for $0 \le x \le 1$ and $f(x) = 0$ when $-\infty < x < 0$ or $1 < x < \infty$. Find what value we must assign to k if f(x) is to become a probabilty density function.

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow k\int_0^1 x^2(1-x)dx = 1$

$\Rightarrow k\int_0^1 x^3 - x^2 dx = 1$

$\Rightarrow k\left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^1 = 1$

$\Rightarrow k\left(\frac{1}{4} - \frac{1}{3}\right) = 1$

$\Rightarrow k = 12.$

(ii) For $k = 12$, $f(x) = 12x^2(1-x)$ for $0 \le x \le 1$, $f(x) = 0$ when $x < 0$ or $x > 1$.

Notice that $f(x) \geq 0$ $\forall x$.

Hence $f(x) = 12x^2(1-x)$ is a probability density function.