NAME:

MA241-006: Calculus II

Instructor: Mr. James Cook
Test: #3 Form A
Date: Monday, March 29, 2006

Directions: You **must** show **ALL** your work to receive credit.

1. (24 pts) Consider the differential equation $\frac{dy}{dx} = y$. If you wish you may set $Dwight = 1$ for convenience.

   a.) Find all equilibrium solution(s).

   b.) Find the *explicit* general solution.

c.) Find the solution that satisfies the initial condition $y(0) = Dwight$.

d.) Find the orthogonal trajectory to the solution found in part c.). The orthogonal trajectory will also satisfy the initial condition $y(0) = Dwight$.

Hint: If you forget how to start, remember the orthogonal trajectory satisfies a differential equation which expresses that it should be locally perpendicular to the original solution. Remember that if a line with slope M is perpendicular to another line with slope m then $M = -1/m$.

2. (16 pts) Find the general solution to the following differential equations. Your solution should be written in terms of real arbitrary constants c_1 and c_2 and real functions (like sine, cosine or exponential of a real argument). In other words, use the characteristic equation to determine the type of solution and then quote the answer we derived in class (case I,II or III). I don't care if you say which case it is, I just want the correct general solution.

a.) $y'' + 5y' + 6y = 0$ (with $y' = \frac{dy}{dx}$)

b.) $y'' = 0$ (with $y' = \frac{dy}{dt}$)

3. (20 pts) Find the solution to $y''+2y'+2y=0$ subject to the boundary conditions $y(0)=1$ and $y(\pi/2)=0$ $\cancel{y(\pi)=0}$. (Thank you Mr. Cook for not giving an initial velocity instead, as the derivative would have been unpleasant) *Reminder: The boundary conditions should fix the values of c_1 and c_2.*

4. (20 pts) Find the general solution of $y'' + y = x + 2e^x$.

5. (15 pts) For each of the following differential equations write the correct particular solution in terms of undetermined coefficients A,B,C,... *do not find specific values for A,B,C,... just set up the correct choice of* Y_p.

a.) $y'' + y = x^3 + 2$

b.) $y'' + y = e^x cos(2x)$

c.) $y'' + y = sin(x) + 3$

6. (5 pts) Consider a mass on a spring (let m=1 and k=1). If the mass is pulled/pushed on by some external force $F_{ext} = 5\ cos(\omega t)$, then Newton's second law yields (assuming no friction and the spring itself has no mass)

$$\frac{d^2x}{dt^2} + x = 5\ cos(\omega t). \tag{1}$$

Explain why the value of ω determines what is the form of the general solution for x(t). Argue that there are two types of solutions that are possible. As in the last problem you may leave the particular part of the solution undetermined, just set it up to point to the general features of the solution. For what special value of ω do we get motion which becomes unbounded (*resonance*) as time goes on ?

MA 241-006, Test III solution, Spring 2006

1.) Let $\frac{dy}{dx} = y$. Then consider,

a.) Equilbrium solutions satisfy $\frac{dy}{dx} = y = 0$ $\Rightarrow$ $\boxed{y=0 \text{ is the equil. sol}^{n}}$

b.) $\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + \tilde{c}$

$\Rightarrow |y| = e^{x+\tilde{c}} = e^{\tilde{c}}e^{x}$

$\Rightarrow y = \pm e^{\tilde{c}}e^{x}$ (let $\pm e^{\tilde{c}} = c$.)

$\Rightarrow \boxed{y = ce^{x}}$

c.) $y(0) = ce^{0} = \text{Dwight} \Rightarrow c = \text{Dwight} \therefore \boxed{y = (\text{Dwight})e^{x}}$

(or $y(0) = ce^{0} = 1 \Rightarrow c = 1 \therefore y = e^{x}$)

d.) The orthogonal trajectories to $\frac{dy}{dx} = y$ are sol^{n}'s to $\frac{dy}{dx} = \frac{-1}{y}$.

$\frac{dy}{dx} = \frac{-1}{y} \Rightarrow y\,dy = -dx$

$\Rightarrow \frac{1}{2}y^2 = c - x$

$\Rightarrow y^2 = 2(c-x)$

$\Rightarrow y = \pm\sqrt{2(c-x)}$

But we need $y(0) = \text{Dwight} = 1 = \pm\sqrt{2(c-0)}$, so we select the + and then

$1 = \sqrt{2c} \Rightarrow 1 = 2c \therefore c = 1/2$

$$\boxed{y = \sqrt{2(1/2 - x)}}$$

② a.) $y'' + 5y' + 6y = 0$ where $dy/dx = y'$.

$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \quad \therefore \quad \lambda = -3$ or -2

$\therefore \boxed{y = c_1 e^{-3x} + c_2 e^{-2x}}$

b.) $y'' = 0$ where $y' = dy/dt$

$\lambda^2 = 0 \Rightarrow \lambda = 0$ or $0 \quad \therefore \quad y = c_1 e^{0t} + t c_2 e^{0 \cdot t} = \boxed{c_1 + t c_2 = y}$

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③ $y'' + 2y' + 2y = 0$ with $y(0) = 1$ and $y(\frac{\pi}{2}) = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \Rightarrow \begin{array}{l} \alpha = -1 \\ \beta = 1 \end{array}$$

Thus, the general sol^n is,

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

But we can go further,

$$y(0) = e^0(c_1 \cos(0) + c_2 \sin(0)) = c_1 = 1$$

$$y(\pi) = e^{-\frac{\pi}{2}}(c_1 \cos(\tfrac{\pi}{2}) + c_2 \sin(\tfrac{\pi}{2})) = e^{-\pi/2} c_2 = 0 \Rightarrow c_2 = 0$$

Thus,

$$\boxed{y = e^{-x} \cos(x)}$$

(4) $y'' + y = x + 2e^x$ find general question.

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i \Rightarrow Y_c = C_1 \cos(x) + C_2 \sin(x)$

Then we guess the form of the particular sol^n is (no overlap)

$$Y_p = Ax + B + Ce^x$$
$$Y_p' = A + Ce^x$$
$$Y_p'' = Ce^x$$

Then substitute

$$Y_p'' + Y_p = x + 2e^x$$

$$Ce^x + Ax + B + Ce^x = x + 2e^x$$

$$2Ce^x + Ax + B = x + 2e^x$$

Comparing coefficients,

$e^x|$ $2C = 2 \Rightarrow C = 1$

$x|$ $A = 1$

$x^0|$ $B = 0$

Thus $Y = Y_c + Y_p = \boxed{C_1 \cos(x) + C_2 \sin(x) + x + e^x = Y}$

(5) Recall $y''+y=0$ has $y_c = c_1\cos(x)+c_2\sin(x)$.

a.) $y''+y = x^3+2 \Rightarrow \boxed{y_p = Ax^3+Bx^2+Cx+D}$

b.) $y''+y = e^x\cos(2x) \Rightarrow \boxed{y_p = Ae^x\cos(2x) + Be^x\sin(2x)}$

c.) $y''+y = \sin(x)+3 \Rightarrow \boxed{y_p = x(A\cos(x)+B\sin(x)) + C}$

In parts a&b no overlap. In part c) the terms from $\sin(x)$ overlap y_c thus multiply by x, but leave C alone as it is already linearly independent from y_c.

(6) $$\frac{d^2x}{dt^2} + x = 5\cos(\omega t)$$

Again this is just $x''+x = 5\cos(\omega t)$ which has

$$x_c(t) = c_1\cos(t) + c_2\sin(t)$$

Then the particular sol^{n} naively would be

$$x_p\ (\text{naive}) = A\cos(\omega t) + B\sin(\omega t)$$

So long as $\omega \neq 1$ there is no overlap. But when $\omega = 1$ there is overlap so

$$x_p = t\,(A\cos t + B\sin t)$$

Thus the motion with $\omega=1$ follows

$$\boxed{x(t) = c_1\cos(t) + c_2\sin(t) + t\,(A\cos t + B\sin t)}$$

In short, $\omega=1$ makes the motion unbounded as $t\to\infty$.