QUIZ 8: POWER SERIES BASICS

§8.5 # 5, 7, 19 & §8.6 # 5, 9, 11, 25

$\boxed{\text{§8.5\#5}}$ Let $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n^3}$.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)^3} \frac{n^3}{(-1)^{n-1} x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| \left| \frac{n^3}{(n+1)^3} \right|$$

$$= |x| \lim_{n\to\infty} \left(\frac{n^3}{n^3+3n^2+3n+1} \right)$$

$$= |x|$$

Therefore $L < 1 \Rightarrow |x| < 1$ and by the ratio test we know I.O.C. includes $-1 < x < 1$. However, the endpoints $x = \pm 1$ we must check seperately,

$\underline{x=1}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$ identify this is an alternating series with $b_n = 1/n^3$ which is clearly positive, continuous and since $\frac{d}{dn}\left(\frac{1}{n^3}\right) = \frac{-3}{n^4} < 0$ ← [extending n to be a cont. variable] we see it's decreasing. Finally since $\frac{1}{n^3} \to 0$ as $n \to \infty$ we conclude the series converges by the A.S.T.

$\underline{x=-1}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n^3} = -\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges by p-series test $p = 3$. To conclude

$\boxed{\text{I.O.C.} = [-1, 1] \text{ and } R = 1 = \text{rad. of convergence}}$

$\boxed{\text{§8.5\#7}}$ Again use ratio test on $\sum_{n=0}^{\infty} x^n/n!$

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right|$$

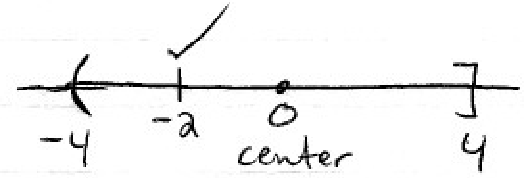
$$= |x| \lim_{n\to\infty} \left(\frac{n!}{(n+1)!} \right)$$ notice $(n+1)! = (n+1)n!$ thus the $n!$ cancels,

$$= |x| \lim_{n\to\infty} \left(\frac{1}{n+1} \right) = 0$$ ($\frac{1}{n+1} \to 0$)

$\therefore \boxed{\text{I.O.C.} = (-\infty, \infty)}$

$\boxed{\S 8.5 \# 19}$ $\sum_{n=0}^{\infty} C_n 4^n$ converges. What can we say about the following? The answers follow from the Th^{m} 3,

(a.) $\sum_{n=0}^{\infty} C_n (-2)^n$



inside the radius of convergence for sure

(b.) $\sum_{n=0}^{\infty} C_n (-4)^n$ this time we can't be sure, it's on the edge.

Remark: $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{x}{4}\right)^{n-1} \frac{1}{n}$ will have the properties the abstract series above has, $f(4)$ converges by A.S.T. whereas $f(-4)$ diverges since its the harmonic ($P=1$) series.

$\boxed{\S 8.6 \# 5}$

$$f(x) = \frac{1}{1-x^3} = \frac{a}{1-r} = a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$$

$$= 1 + x^3 + x^6 + x^9 + \cdots = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}.$$

$\boxed{\S 8.6 \# 9}$ Again use geom. series with $a = 1$, $r = -x^2/9$.

$$f(x) = \frac{x}{9+x^2} = \frac{x}{9}\left(\frac{1}{1+x^2/9}\right) \text{ : factored out } \frac{x}{9}$$

$$= \frac{x}{9}\left(1 - \frac{x^2}{9} + \left(\frac{x^2}{9}\right)^2 - \left(\frac{x^2}{9}\right)^3 + \cdots\right)$$

$$= \frac{x}{9} - \frac{x^3}{81} + \frac{x^5}{729} - \frac{x^7}{9(729)} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{9^{n+1}} x^{2n+1}$$

$\boxed{\S 8.6 \ \#11}$ $f(x) = \dfrac{1}{(1+x)^2}$

(a.) $\displaystyle\int f\,dx = \frac{-1}{1+x} + C$

geom. series with
$a = -1$
$r = -x$

$= -1(1 - x + x^2 - x^3 \ldots) + C$

$$f(x) = \frac{d}{dx}\int f\,dx = \boxed{1 - 2x + 3x^2 - \cdots = \frac{1}{(1+x)^2}}$$

(b.) $g(x) = \dfrac{1}{(1+x)^3}$

$\displaystyle\int g\,dx = \frac{-1}{2}\,\frac{1}{(1+x)^2} + C$

$\displaystyle = \frac{-1}{2}\left(1 - 2x + 3x^2 - \cdots\right) + C$

$$g(x) = \frac{d}{dx}\int g\,dx = \frac{d}{dx}\left(\frac{-1}{2} + x - \frac{3}{2}x^2 + \cdots + C_0\right)$$

$$= \boxed{1 - x + \cdots + \cdots = g(x) = \frac{1}{(1+x)^3}}$$

(c.) $\displaystyle\frac{x^2}{(1+x)^3} = x^2(1 - x + \cdots) = \boxed{x^2 - x^3 + \cdots = \frac{x^2}{(1+x)^3}}$

Now its better to use Σ-notation since we get more complete answers, concisely the arguments become,

(a.) $\displaystyle\frac{1}{(1+x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} -(-x)^n = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$

(b.) $\displaystyle\frac{1}{(1+x)^3} = \frac{d}{dx}\,\frac{-1}{2}\sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \sum_{n=2}^{\infty}\frac{(-1)^{n+2}}{2}\,n(n-1)x^{n-2}$

(c.) $\displaystyle\frac{x^2}{(1+x)^3} = x^2\sum_{n=2}^{\infty}(-1)^{n+2}\frac{1}{2}n(n-1)x^{n-2} = \sum_{n=2}^{\infty}(-1)^{n+2}\frac{1}{2}n(n-1)x^{n}$

$\boxed{\S 8.6 \# 25}$ use geometric series to convert $\frac{1}{1+x^5}$ to pow. series,

$$\int \frac{1}{1+x^5}\,dx = \int (1 - x^5 + x^{10} - x^{15} + \cdots)\,dx$$

$$= x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots + C$$

Now the definite integral will reveal the answer is an alternating series!

$$\int_0^{0.2} \frac{dx}{1+x^5} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \cdots$$

It's quite clear $(0.2)^{11}/11$ is several decimals beyond the 6^{th} decimal and by the Alt. Series Estimation Th^{m} we know that

$$\int_0^{0.2} \frac{dx}{1+x^5} = 0.2 - \frac{(0.2)^6}{6} \quad \text{within} \quad \frac{(0.2)^{11}}{11}$$

$$\cong \boxed{0.199989}$$

(could do even better given $\frac{(0.2)^{11}}{11}$ is actually beyond the 6^{th} decimal.)