YOU MAY KEEP THE TEST PAPER, PUT NAME ON WORK.

Instructor: James Cook , MA 241-003 Calculus II, Test III: Differential Equations, July 23, 2007

Directions: Show your work, if you doubt that you've shown enough detail then ask. No electronic aids of any sort are permitted modulo your watch and a scientific calculator.

1. (5pts) Consider the following differential equation,
$$\left(\frac{dy}{dx}\right)^2 + y^2 = 1$$
show that $y = \sin(x)$ is a solution. For a bonus point find another solution by guessing and then show that the sum of solutions is not a solution for this nonlinear differential equation.

2. (15pts) Find the solution to
$$\frac{dy}{dx} = \frac{x+3}{y}$$
such that $y(0) = -7$.

3. (15pts) Solve the following differential equation using the integrating factor method,
$$\frac{dy}{dx} - y = e^x$$

4. (15pts) Solve the following differential equations,
(a.) $y'' + 4y' + 5y = 0$
(b.) $y'' - 5y' + 6y = 0$
(c.) $y'' + 4y' + 4y = 0$

5. (25pts) Solve the following, use the method of undetermined coefficients to find y_p,
$$y'' + y = x^2 + 2e^x$$

6. (15pts) Solve the following, use the method of undetermined coefficients to find y_p,
$$y'' + y' = 3$$

7. (10pts) Solve the following using variation of parameters,
$$y'' + y = \tan(x)$$
recall that $\int \sec(x)dx = \ln|\sec(x) + \tan(x)|$.

8. BONUS:(15pts) Calculate the following,
(5pts): Solve problem 3 using the method of undetermined coefficients
(5pts): $\int \sin(ax)\cos(3x)\, dx$,
(5pts): $\int \frac{1}{\sin(x) + 2\cos(x)}\, dx$,
WARNING: don't even try this last one unless you have everything else completely finished and checked, this one is nontrivial.

Sol$^{\underline{n}}$ to ma 241-003 summer II test III (2007)

$\boxed{\text{PROBLEM ONE}}$ Show $y = \sin(x)$ solves $(y')^2 + y^2 = 1$,

$$\frac{d}{dx}(\sin(x))^2 + (\sin(x))^2 = \cos^2(x) + \sin^2(x) = 1.$$

Notice $y = \cos(x)$ is also a sol$^{\underline{n}}$ since $(-\sin(x))^2 + \cos^2 x = 1$.
However $y = \sin(x) + \cos(x)$ is not a sol$^{\underline{n}}$ since

$$\begin{aligned}(y')^2 &= (\cos(x) - \sin(x))^2 = \cos^2 x - 2\sin x\cos x + \sin^2 x\\ + \quad y^2 &= (\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x\\ \hline (y')^2 + y^2 &= 2\cos^2 x + 2\sin^2 x = 2 \neq 1.\end{aligned}$$

We used ~~that~~ the fact that y_1 & y_2 sol$^{\underline{n}}$'s $\Rightarrow y_1 + y_2$ is also a sol$^{\underline{n}}$ throughout our work on linear ODEs.

$\boxed{\text{PROBLEM TWO}}$ We can use seperation of variables,

$$\frac{dy}{dx} = \frac{x+3}{y} \Rightarrow \int y\,dy = \int (x+3)\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + 3x + C$$

$$\Rightarrow y^2 = x^2 + 6x + 2C \qquad \text{call } 2C = C_2$$

$$\Rightarrow y = \pm\sqrt{x^2 + 6x + C_2}$$

the $(\pm)$ are crucial here since if you chose $(+)$ w/o acknowledging the $(-)$ you missed the sol$^{\underline{n}}$ with $y(0) = -7$.

$$-7 = -\sqrt{0 + 6(0) + C_2} \Rightarrow -7 = -\sqrt{C_2}$$

$$\Rightarrow 49 = C_2$$

$$\therefore \boxed{y = -\sqrt{x^2 + 6x + 49}}$$

PROBLEM 3) Use int. factor method to solve

$$\frac{dy}{dx} - y = e^x$$

① $I = \exp\left(\int -dx\right) = \exp(-x) = e^{-x}$.

② multiply by I,

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{x} = 1.$$

③ product rule, $\frac{d}{dx}\left(e^{-x}y\right) = 1$

④ integrate, $e^{-x}y = x + C$

multiply by e^x, $\boxed{y = xe^x + Ce^x.}$

PROBLEM 4)

(a.) $y'' + 4y' + 5y = 0$

$\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm \text{[redacted]}\, i$

$\therefore \boxed{Y = C_1 e^{-2x}\cos(\text{[redacted]}x) + C_2 e^{-2x}\sin(\text{[redacted]}x)}$

(oops.)

(b.) $y'' - 5y' + 6y = 0$

$\lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0 \quad \therefore \lambda_1 = 3,\ \lambda_2 = 2$

$\therefore \boxed{Y = C_1 e^{3x} + C_2 e^{2x}}$

(c.) $y'' + 4y' + 4y = 0$

$\lambda^2 + 4\lambda + 4 = (\lambda + 2)(\lambda + 2) \quad \therefore \lambda_1 = -2 = \lambda_2$

$\therefore \boxed{Y = C_1 e^{-2x} + C_2 x e^{-2x}}$

(Where I have used the results derived in lecture on all of the above)

$\boxed{\text{PROBLEM 5}}$ Solve $y'' + y = x^2 + 2e^x$. To begin notice $\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i \therefore y_h = c_1 \cos x + c_2 \sin x$.

Guess
$$y_p = Ax^2 + Bx + C + De^x \quad \text{, no overlap.}$$
$$y_p' = 2Ax + B + De^x$$
$$y_p'' = 2A + De^x$$

$$y_p'' + y_p = x^2 + 2e^x$$
$$2A + De^x + Ax^2 + Bx + C + De^x = x^2 + 2e^{2x}$$

<u>Equate Coefficients</u>:

$\underline{\text{const}}$: $2A + C = 0 \quad \Rightarrow C = -2A = \underline{-2 = C}$

$\underline{x}$: $\underline{B = 0}$

$\underline{x^2}$: $\underline{A = 1}$

$\underline{e^x}$: $D + D = 2 \quad \Rightarrow 2D = 2 \Rightarrow \underline{D = 1}$

Thus in total we find,
$$\boxed{y = c_1 \cos(x) + c_2 \sin(x) + x^2 - 2 + e^x}$$

PROBLEM 6 Solve $y'' + y' = 3$. Notice to begin
$\lambda^2 + \lambda = \lambda(\lambda+1) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = -1 \Rightarrow y_h = C_1 + C_2 e^{-x}$
Guess $Y_p = A$ but this overlaps with $\lambda_1 = 0$ so multiply Y_p by x and try $Y_p = Ax$ instead,

$$\left.\begin{aligned} Y_p &= Ax \\ Y_p' &= A \\ Y_p'' &= 0 \end{aligned}\right\} \qquad \begin{aligned} y_p'' + y_p' &= 3 \\ A &= 3 \end{aligned}$$

Thus $\boxed{Y = C_1 + C_2 e^{-x} + 3x}$

PROBLEM 7 Solve $y'' + y = \tan(x)$.
$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow \underline{y_1 = \cos(x)}$ & $\underline{y_2 = \sin(x)}$
So we may calculate $y_1' y_2 - y_1 y_2' = -\sin^2 x - \cos^2 x = -1$ so, the denominator in the formula for V_1 or V_2 is simple.
Again we guess $y_p = V_1 \cos(x) + V_2 \sin(x)$ and recall that V_1 & V_2 may be found via

$$V_1 = \int -y_2 g\,dx = \int -\sin(x)\frac{\sin x}{\cos x}\,dx = \int \frac{\cos^2 x - 1}{\cos x}\,dx = \int (\cos x - \sec x)\,dx$$

$$V_2 = \int y_1 g\,dx = \int \cos x \frac{\sin x}{\cos x}\,dx = \int \sin x\,dx$$

Thus $Y_p = (\sin x - \ln|\sec x + \tan x|)\cos x + (-\cos x)\sin x$.
Finally
$$Y = Y_h + Y_p = \boxed{C_1 \cos x + C_2 \sin x - (\ln|\sec x + \tan x|)\cos x = Y}$$

Bonus: Solve $\frac{dy}{dx} - y = e^x$ using undet. coeff.
I didn't mention this but it works the same,

$$\lambda - 1 = 0 \Rightarrow \lambda = 1 \quad \therefore \quad y_h = c_1 e^x$$

Now $g(x) = e^x \Rightarrow$ naively $y_p = Ae^x$ but there's overlap so try $y_p = Axe^x$ instead,

$$y_p' = Ae^x + Axe^x$$

thus,

$$y_p' - y_p = e^x$$

$$Ae^x + \cancel{Axe^x} - \cancel{Axe^x} = e^x$$

$$Ae^x = e^x$$

$$\therefore A = 1 \qquad \therefore \quad y = y_h + y_p$$

$$\boxed{y = c_1 e^x + xe^x}$$

- this is the same as we found by the integrating factor method.

- the 5pt. bonus can be solved by $t = \tan(x/2)$.
- the other 5pt. bonus can be solved by using $e^{i\theta} = \cos\theta + i\sin\theta$ tricks.