YOU MAY KEEP THE TEST PAPER, PUT NAME ON WORK.

Instructor: James Cook , MA 241-003 Calculus II, Test IV: Sequences and Series, July 30, 2007

Directions: Show your work, if you doubt that you've shown enough detail then ask. No electronic aids of any sort are permitted modulo your watch and a scientific calculator.

1. (6pts) Determine the limit of the sequence below. Give an argument to explain why you are correct, this could be algebra, a graph, maybe L'Hopital's Rule, numerical evidence, it's up to you.

$$\lim_{n\to\infty} \frac{2n^2+6}{n+3n^2}$$

2. (9pts) A series is the limit of the sequence of partial sums. Find the n^{th} partial sum then take the limit to prove the following telescoping series converges to 3/2,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{3}{2}$$

3. (20pts) Do the series below converge or diverge ? If possible calculate the value to which the series converges. Explain which test you used, check any needed criteria,

(a.) $s = 1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$

(b.) $\sum_{n=1}^{\infty} \frac{1}{n}$

(c.) $\sum_{n=1}^{\infty} \tan^{-1}(n)$

(d.) $\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n^2}$

4. (4pts) We'll soon learn that $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$. Thus if we evaluate this power series at x=1 we find,

$$sin(1) = 1 - \frac{1}{6} + \frac{1}{120} + \cdots$$

Use the alternating series estimation theorem to say what the maximum error in the approximation $\sin(1) \sim 1 - \frac{1}{6} = \frac{5}{6}$.

5. (16pts) Find the Interval Of Convergence (I.O.C.) and radius of convergence for the following power series,

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{3}\right)^{n-1}$$

Hint: use ratio test, don't forget to check the endpoints.

6. (40pts) Use the geometric series to find power series representations for the functions below, please give the first three nonzero terms (no need for the Σ notation here, you can use the $+\cdots$ notation for this problem)

(a.) $f(x) = \dfrac{1}{1-x}$

(b.) $g(x) = \dfrac{x}{2-x^5}$

(c.) $h(x) = \ln(1+x)$

(d.) $j(x) = \dfrac{1}{(1+x)^2}$

7. (10pts) Find the complete power series solution to the following integral, for full credit please give the answer and analysis in terms of the Σ notation (if you give me the solution correctly in the $+\cdots$ notation I will deduct 3 pts)

$$\int \frac{x}{2-x^5}\,dx$$

keep in mind part (b.) of the previous problem.

Ma 241 – TEST IV Sol^n – SUMMER II – 2007

PROBLEM ONE $\displaystyle\lim_{n\to\infty}\left(\frac{2n^2+6}{n+3n^2}\right) = \lim_{n\to\infty}\left(\frac{2+\cancel{6/n^2}^{\,0}}{\cancel{1/n}_{\,0}+3}\right) = \boxed{\frac{2}{3}}$

other sol^n's are of course possible here.

PROBLEM TWO

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right)$$

$$= \left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\cdots$$

$$+\cdots+\left(\cancel{\tfrac{1}{n-3}}-\cancel{\tfrac{1}{n-1}}\right)+\left(\cancel{\tfrac{1}{n-2}}-\cancel{\tfrac{1}{n}}\right)+\left(\cancel{\tfrac{1}{n-1}}-\tfrac{1}{n+1}\right)+\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+2}\right)$$

$$\Rightarrow S_n = 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$S = \lim_{n\to\infty}\left(1+\frac{1}{2}-\underbrace{\frac{1}{n+1}}_{0}-\underbrace{\frac{1}{n+2}}_{0}\right) = \boxed{\frac{3}{2} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)}$$

PROBLEM THREE

(a.) $\displaystyle S = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$

this series is geometric with $r=\frac{1}{2}$ so it converges (to 2).

(b.) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, p-series with $p=1$ $\therefore$ diverges.

(c.) $\displaystyle\sum_{n=1}^{\infty}\tan^{-1}(n)$, note $a_n = \tan^{-1}(n) \longrightarrow \frac{\pi}{2}$ as $n\to\infty$

thus the series diverges as $\displaystyle\lim_{n\to\infty} a_n \neq 0$. ($n^{th}$ term divergence test)

(d.) could use A.S.T. or ratio test to show converges.

PROBLEM THREE

(d.) Sol^n I (Ratio Test)

$$L = \lim_{n\to\infty} \left| \frac{(-1)^n e^{-(n+1)^2}}{(-1)^{n-1} e^{-n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{e^{-n^2 - 2n - 1}}{e^{-n^2}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{1}{e^{2n+1}} \right) = 0 < 1 \quad \therefore \text{ series converges by ratio test.}$$

Sol^n II. (Alternating Series Test)

Observe $\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n^2}$ is alternating with $b_n = e^{-n^2}$

clearly $b_n = e^{-n^2} > 0$ and $e^{-n^2} \longrightarrow 0$ as $n \longrightarrow \infty$.
Moreover it's decreasing since (extending n to be continuous)

$$\frac{d}{dn}\left(e^{-n^2}\right) = -2ne^{-n^2} < 0 \quad \text{for } n \geq 1$$

therefore by A.S.T. this series converges.

PROBLEM FOUR Observe that $\sin(1) = 1 - \frac{1}{6} + \frac{1}{120} - \cdots$ is an alternating series $\therefore$ $\sin(1) \cong 5/6$ to within $1/120$ of the exact value of $\sin(1)$ by the alternating series estimation theorem.

$\boxed{\text{PROBLEM FIVE}}$ Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{3}\right)^{n-1}$.

$$L = \lim_{n\to\infty} \left| \frac{(x/3)^n}{n+1} \frac{n}{(x/3)^{n-1}} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{x}{3}\right)\left(\frac{n}{n+1}\right) \right|$$

$$= \left|\frac{x}{3}\right| \underbrace{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)}_{\to 1}$$

$$= \left|\frac{x}{3}\right| < 1 \Rightarrow -1 < \frac{x}{3} < 1 \Rightarrow -3 < x < 3$$

will converge
$x = \pm 3$ don't know yet.

$f(3) = \sum_{n=1}^{\infty} \frac{1}{n}$, p-series $p=1 \Rightarrow$ diverges.

$f(-3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$, $b_n = \frac{1}{n} > 0$, $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$

and $\frac{1}{n} \to 0$ as $n \to \infty$ $\therefore$ converges by A.S.T.

$\boxed{\text{So we find I.O.C} = [-3, 3) \text{ and } R = 3}$

$\boxed{\text{PROBLEM SIX}}$

(a.) $f(x) = \dfrac{1}{1-x} = \boxed{1 + x + x^2 + \cdots} = \sum_{n=0}^{\infty} x^n.$

(here $a = 1$ and $r = x$)

(b.) $g(x) = \dfrac{x}{2-x^5} = \dfrac{x/2}{1 - x^5/2}$ $\quad a = \frac{x}{2}$ & $r = \frac{x^5}{2}$

$= \frac{x}{2} + \frac{x}{2}\frac{x^5}{2} + \frac{x}{2}\left(\frac{x^5}{2}\right)^2 + \cdots$

$= \boxed{\frac{1}{2}x + \frac{1}{4}x^6 + \frac{1}{8}x^{11} + \cdots} = \sum_{n=0}^{\infty} \frac{x}{2}\left(\frac{x^5}{2}\right)^n$

$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^{5n+1}$

(c.) $h(x) = \ln(1+x)$

$h'(x) = \dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

$h(x) = \int h'(x)\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + C = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$

$h(0) = \ln(1+0) = \ln(1) = C \therefore C = 0$

$\therefore \boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots}$

(d.) $\int j(x)\,dx = \int \frac{dx}{(1+x)^2} = \int \frac{du}{u^2} = \frac{-1}{u} + C = \frac{-1}{x+1} + C$

$\int j(x)\,dx = -1 + x - x^2 + x^3 - \cdots \quad (a = -1,\ r = -x)$

$j(x) = \frac{d}{dx}\int j(x)\,dx = \frac{d}{dx}\left(-1 + x - x^2 + x^3 - \cdots\right)$

$\boxed{j(x) = 1 - 2x + 3x^2 + \cdots}$

PROBLEM SEVEN picking up where we left off in (6 b)

$$\int \frac{x}{2-x^5}\,dx = \int\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\, x^{5n+1}\right)dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int x^{5n+1}\,dx$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\,\frac{x^{5n+2}}{5n+2} + C}$$ (for 10pts)

$$= \frac{1}{4}x^2 + \frac{1}{28}x^7 + \frac{1}{96}x^{12} + \cdots + C$$ (for 7pts)

Comment: for problem THREE the biggest and most common conceptual error was not distinguishing between the convergence of a_n and the convergence of $\sum a_n$. These are related but quite different.

- future classes should keep in mind we had no Taylor or Binomial Series to employ. Only geometric series tricks for this test.