NAME:

MA 241.07 TEST III

APRIL 1, 2005

1 Problem One

In this problem consider the differential equation $\frac{dy}{dx} = e^{x+y}$.

1a.(10pts) solve $\frac{dy}{dx} = e^{x+y}$ explicitly.

$$\frac{dy}{dx} = e^{x+y} = e^x e^y \implies e^{-y}\,dy = e^x\,dx$$: Seperate variables

$$\implies -e^{-y} = e^x + C_1$$: integrate

$$\implies e^{-y} = C_2 - e^x$$: make both sides positive

$$\implies -y = \ln(C_2 - e^x)$$: take ln

$$\implies y = -\ln(C_2 - e^x)$$: solve for y.

$$\implies \boxed{y = \ln\left(\frac{1}{C - e^x}\right)}$$

1b.(10pts) find the orthogonal trajectories of $\frac{dy}{dx} = e^{x+y}$.

To find O.Ts we must solve $\frac{dy}{dx} = \frac{-1}{e^{x+y}} = -e^{-x}e^{-y}$ that is sep. variables,

$$-e^{y}\,dy = -e^{-x}\,dx \Rightarrow -e^{y} = -(-e^{-x}) + k \leftarrow$$

one (−) from the Deqⁿ itself and one (−) from $\int e^{-x}dx = -e^{-x} + C$.

$$\Rightarrow \boxed{y = \ln(e^{-x} + k)}$$

1c. (10pts)Find the equations of the curve and orthogonal trajectory that intersect at the origin $(0, 0)$. That is find specific solutions of 1a and 1b that intersect at the origin.

Solⁿ has $y(0) = 0 \Rightarrow \ln\left(\frac{1}{c - e^{0}}\right) = 0$

$$\Rightarrow \ln\left(\frac{1}{c-1}\right) = 0$$

$$\Rightarrow \frac{1}{c-1} = 1$$

$$\Rightarrow 1 = c - 1 \Rightarrow \boxed{c = 2}$$

O.T. has $y(0) = 0$ also thus $\Rightarrow \ln(k + e^{-0}) = 0$

$$\Rightarrow \ln(k+1) = 0$$

$$\Rightarrow k + 1 = 1$$

$$\Rightarrow \boxed{k = 0}$$

Solⁿ thru $(0,0)$ is $\boxed{y = \ln\left(\frac{1}{2 - e^{x}}\right)}$ with orthogonal trajectory $\boxed{y = \ln(e^{-x}) = -x}$, as a check you can verify that at the origin these fncts ⊥ slopes at $x = 0$,

$$\frac{d}{dx}(-x) = -1 \quad \& \quad \frac{d}{dx}\left(\ln(2 - e^{-x})\right)\Big|_{x=0} = 1$$

(they have ⊥ tangents at $(0,0)$).

2 Problem Two

In this problem consider the differential equation $\frac{dy}{dx} = \frac{\sin(x)}{y^4+y^2-23}$.

2a.(10pts) find all equilibrium solutions, use an integer n to list the answers.

$\frac{\sin(x)}{y^4+y^2-23} = 0$ only when $\sin(x) = 0$

and we know $\sin(x) = 0$ when $\boxed{x = n\pi, \ n \in \mathbb{Z}.}$

$(x = \ldots, -\pi, 0, \pi, 2\pi, \ldots)$

2b.(10pts) find an implicit solution to the differential equation.

$\frac{dy}{dx} = \frac{\sin(x)}{y^4+y^2-23} \Rightarrow (y^4+y^2-23)dy = \sin(x)dx$

$\Rightarrow \boxed{\frac{1}{5}y^5 + \frac{1}{3}y^3 - 23y = -\cos(x) + C}$

this is an implicit sol^{n}.

3 Problem Three

3.(20pts) Find the general solution of $y'' + 3y = e^x \sin(x)$. Your answer should involve only 2 unknown constants.

Homogeneous sol^{n}: Consider $\lambda^2 + 3 = 0 \Rightarrow \lambda = \pm i\sqrt{3}$

thus $Y_H = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)$.

Particular sol^{n}: Clearly no overlap so guess

$$Y_p = e^x (A\cos(x) + B\sin(x))$$

$$Y_p' = e^x(A\cos(x) + B\sin(x)) + e^x(-A\sin(x) + B\cos(x))$$
$$= e^x((A+B)\cos(x) + (B-A)\sin(x))$$

$$Y_p'' = e^x((A+B)\cos(x) + (B-A)\sin(x)) + e^x(-(A+B)\sin(x) + (B-A)\cos(x))$$
$$= e^x((A+B+B-A)\cos(x) + (B-A-A-B)\sin(x))$$
$$= e^x(2B\cos(x) - 2A\sin(x))$$

Now determine coefficients A & B by subst. into $Y_p'' + 3Y_p = e^x\sin(x)$

$$Y_p'' + 3Y_p = e^x(2B\cos(x) - 2A\sin(x)) + 3e^x(A\cos(x) + B\sin(x))$$
$$= e^x((2B+3A)\cos(x) + (3B-2A)\sin(x)) = e^x\sin(x)$$

Equate Coefficients:

$$2B + 3A = 0 \Longrightarrow A = -\tfrac{2}{3}B$$
$$3B - 2A = 1 \Longrightarrow 3B - 2(-\tfrac{2}{3}B) = 1 \Rightarrow 9B + 4B = 3$$
$$\Rightarrow B = \tfrac{3}{13}$$

Thus the general sol^{n} is

$$\Rightarrow A = -\tfrac{2}{13}$$

$$\boxed{Y = C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x) + e^x\left(\tfrac{-2}{13}\cos(x) + \tfrac{3}{13}\sin(x)\right)}$$

4 Problem Four

Find the homogeneous (a.k.a. complementary in your text) solution to each of the following differential equations(5pts each). Then for each set-up the form of the particular solution using undetermined coefficients(5pts each). DO NOT DETERMINE THE COEFFICIENTS , please.

4a.(10pts) $y'' - y = \sin(x) + x^3$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow \boxed{Y_H = C_1 e^{x} + C_2 e^{-x}}$$

No over lap thus I make

$$\boxed{Y_P = A\sin(x) + B\cos(x) + Cx^3 + Dx^2 + Ex}$$

(You could add F onto Y_P but it would be zero. once you work it out.)

4b.(10pts) $y'' + 2y' + y = xe^{-x}$

$$\lambda^2 + 2\lambda + 1 = (\lambda+1)(\lambda+1) = 0 \Rightarrow \boxed{Y_H = C_1 e^{-x} + C_2 x e^{-x}}$$

Ordinary guess for Y_P would be $Axe^{-x} + Be^{-x}$ but there is a double over lap so I must instead use

$$Y_P = x^2(Axe^{-x} + Be^{-x})$$

4c.(10pts) $\frac{d^2y}{dt^2} + 4y = t + \sin(2t)$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

$$\Rightarrow \boxed{Y_H = C_1 \cos(2t) + C_2 \sin(2t)}$$

There is overlap again with the $\sin(2t)$ thus,

$$\boxed{Y_P = At + B + t(A\sin(2t) + B\cos(2t))}$$

Notice that. you can treat t and $\sin(2t)$ as seperate problems, then just add the result together.

- Note you should only multiply the over lapping part by t.