NAME:

ALIAS:

BONUS POINTS: (list each & when & why)

MA 241.07 TEST IV

APRIL 22, 2005

REMEMBER, "RIGOR TO MATHEMATICIANS IS WHAT MORALITY IS TO MAN"

1.(10pts) Prove that $a + ar + ar^2 + ar^3 + ... = \frac{a}{1-r}$ when $|r| < 1$.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$- S_n r = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$
$$S_n(1-r) = a - ar^n$$
$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Then recall $a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty}\left(\frac{a}{1-r}\right) - \lim_{n\to\infty}\left(\frac{ar^n}{1-r}\right)$ $\to 0$

Where we needed $|r| < 1$ for the 2^{nd} limit to vanish, hence

$$\boxed{a + ar + ar^2 + \cdots = \frac{a}{1-r}}$$

Since the limit of a constant sequence is the constant $\frac{a}{1-r}$.

2.(30pts) Use the divergence/convergence tests we developed in lecture to prove that the following series converge or diverge. Credit will be assigned according to the quality of your answers. (Choose 3)

(a.) $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$ we compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ which is $P=1$ series (diverges)

Via the limit comparison test,

$$\lim_{n\to\infty}\left(\frac{\left(\frac{n}{n^2+3}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2+3}\right) = 1 > 0$$

Thus by limit comp. test we know $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(b.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln n}{n}$ this is an alternating series with $b_n = \frac{\ln(n)}{n}$.

Notice that $b_n > 0$ for $n \geq 1$ and consider then (extending n contin)

$$\frac{d}{dn}\left(\frac{\ln(n)}{n}\right) = \frac{1}{n^2} - \frac{\ln(n)}{n^2} = \frac{1}{n^2}(1 - \ln(n)) \leq 0 \text{ for } n \geq 1$$

since $\ln(1) = 0$ whereas $\ln(n) > 0$ for $n > 1$. So we have that the derivative is negative $\Rightarrow$ $b_{n+1} \leq b_n$. Finally note

$$\lim_{n\to\infty}(b_n) = \lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) \stackrel{\text{ℓ}}{=} \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{1}\right) = 0$$ as req^d by

the alternating series test. Thus b_n are positive, decreasing and have limit zero as $n \to \infty$ $\therefore$ $\boxed{\sum \frac{(-1)^{n+1}\ln(n)}{n} \text{ converges}}$

(c.) $\sum_{n=0}^{\infty} \frac{e^n}{\pi^n} = 1 + \left(\frac{e}{\pi}\right) + \left(\frac{e}{\pi}\right)^2 + \cdots$ geometric series

with $a = 1$ & $r = \frac{e}{\pi} \cong \frac{2.718}{3.141} < 1$ thus it converges.

to $\frac{a}{1-r} = \frac{1}{1 - e/\pi}$ as we proved in #1 in general.

(d.) $\sum_{n=1}^{\infty} \frac{n+3}{3n-2}$ Simplest of all just use n^{th} term test

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\frac{n+3}{3n-2}\right) = \frac{1}{3} \neq 0$$

Therefore by n^{th} term test $\sum_{n=1}^{\infty} \frac{n+3}{3n-2}$ diverges

3.(30pts) Find the first three non-zero terms in the power series expansion centered about zero ($a = 0$) for the following functions:

(a.) $\frac{\sin(x)}{x}$ Recall $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

then simply multiply by $\frac{1}{x}$:

$$\frac{\sin(x)}{x} = \frac{1}{x}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

$$= \boxed{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots}$$

(c.) $\cos^2(x)$ Recall $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots$

$$\cos^2(x) = \cos(x) \cdot \cos(x)$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{48}x^6 + \frac{1}{24}x^4 - \frac{1}{48}x^6 + \frac{1}{(24)^2}x^8$$

$$= 1 - x^2 + \left(\frac{1}{24} + \frac{1}{4} + \frac{1}{24}\right)x^4 + \cdots$$ (just need 1st 3 terms)

$$= \boxed{1 - x^2 + \frac{1}{3}x^4 + \cdots}$$

Similarly you could have done

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$= \frac{1}{2}\left(1 + 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 + \cdots\right) \qquad \frac{2^4}{4!} = \frac{16}{24}$$

$$= \frac{1}{2}\left(2 - 2x^2 + \frac{2}{3}x^4 + \cdots\right)$$

$$= \boxed{1 - x^2 + \frac{1}{3}x^4 + \cdots}$$

it's all good.

Two methods are possible (well more actually)

(b.) $\frac{x^2}{(1-2x)^2} = x^2(1-2x)^{-2}$ use binomial series on $(1-2x)^{-2}$ is one way to do it

$$(1-2x)^{-2} = (1+u)^k \quad \text{with } \begin{array}{l} u=-2x \\ k=-2 \end{array}$$

$$= 1 + ku + \tfrac{1}{2}k(k-1)u^2 + \cdots$$

$$= 1 - 2(-2x) + \tfrac{1}{2}(-2)(-3)(-2x)^2 + \cdots$$

$$= 1 + 4x + 12x^2 + \cdots$$

Hence $\frac{x^2}{(1-2x)^2} = x^2(1+4x+12x^2+\cdots) = \boxed{x^2 + 4x^3 + 12x^4 + \cdots}$

The other method is to change it to a geometric series by integration like in §8.6 — $u = 1-2x$, $du = -2dx$

$$\int \frac{1}{(1-2x)^2}\,dx = \int \frac{1}{u^2}\frac{du}{-2} = \frac{-1}{u}\frac{-1}{2} + C = \frac{1}{2(1-2x)} + C$$

$$\int \frac{1}{(1-2x)^2}\,dx = \frac{1/2}{1-2x} + C = \tfrac{1}{2}(1 + 2x + (2x)^2 + (2x)^3 + \cdots) + C$$ geometric series $a = 1/2$, $r = 2x$

Then differentiate & use FTC on LHS while using term by term diff on RHS

$$\frac{1}{(1-2x)^2} = \tfrac{1}{2}(2 + 2(2x)\cdot 2 + 3(2x)^2\cdot 2 + \cdots) = 1 + 4x + 12x^2 + \cdots$$

Thus $\frac{x^2}{(1-2x)^2} = x^2(1 + 4x + 12x^2 + \cdots)$

$= \boxed{x^2 + 4x^3 + 12x^4 + \cdots}$

4.(20pts) Find the radius of convergence (R) and the interval of covergence (I.O.C.) for the following power series.

(a.) $\sum 2^n(x+1)^n$ this is a geometric series with $a = 1$ & $r = 2(x+1)$

it converges iff $|r| < 1 \Rightarrow |2(x+1)| < 1$

$$|x+1| < \frac{1}{2}$$

Thus the $R = \frac{1}{2}$ and $-\frac{3}{2} < x < -\frac{1}{2}$ is the I.O.C.

(b.) $\sum \frac{(-1)^n(2x-4)^n}{n}$ Use ratio test

$$L = \lim_{n\to\infty}\left|\frac{(2x-4)^{n+1}}{n+1}\frac{n}{(2x-4)^n}\right| = |2x-4| \lim_{n\to\infty}\underbrace{\cancel{\left(\frac{n}{n+1}\right)}}_{\to 1} = |2x-4|$$

Then $L < 1 \Leftrightarrow |2x-4| < 1 \Rightarrow |x-2| < \frac{1}{2} \Rightarrow \boxed{R = \frac{1}{2}}$

The endpoints need to be checked

$$x = 2 - \frac{1}{2} \rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n} = \sum\frac{1}{n} \text{ diverges, } (P=1)$$

$$x = 2 + \frac{1}{2} \rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n(1)^n}{n} = \sum\frac{(-1)^n}{n}$$ converges by alt. series test as $\frac{1}{n}$ is positive, decreasi[ng] and has limit zero as $n\to\infty$.

Hence

$$\text{I.O.C} = \left(\frac{3}{2}, \frac{5}{2}\right]$$

aka $\frac{3}{2} < x \leq \frac{5}{2}$ is the interval of convergence.

5.(10pts) Calculate the integral below. Give the answer as a power series in "sigma notation".

$\int \frac{\sin(x)}{x}\, dx \quad = \int \frac{1}{x}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) dx$

$= \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\right) dx$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int x^{2n}\, dx + C$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{(2n+1)} + C}$