Cartesian Coordinates

We begin by settling some notations to be used throughout the course,

$$\mathbb{R} \equiv (-\infty, \infty)$$

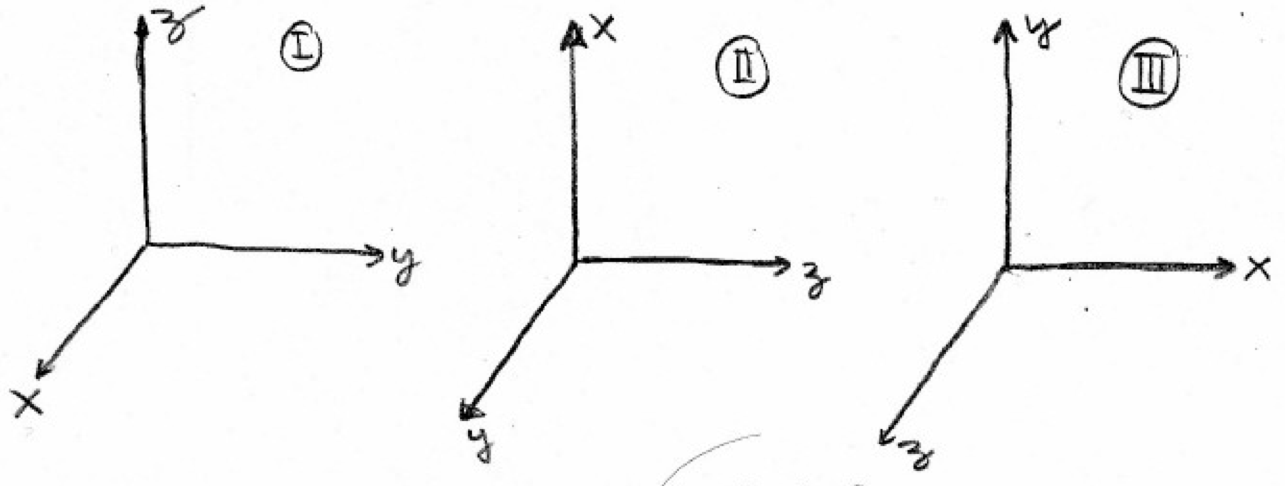
$$\mathbb{R}^2 \equiv \{ (x,y) \mid x, y \in \mathbb{R} \} = \mathbb{R} \times \mathbb{R}$$

$$\mathbb{R}^3 \equiv \{ (x,y,z) \mid x,y,z \in \mathbb{R} \} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

"Cartesian Products"

$$\mathbb{R}^n \equiv \{ (x_1, x_2, \dots, x_n) \mid n \text{ a positive integer}, \ x_i \in \mathbb{R} \ \ i=1,2,\dots,n \}$$

An element of $\mathbb{R}^n$ is called an "n-tuple" or a "point". In the cases of $n = 1, 2$ or 3 we can identify these spaces with our everyday motions where height, width and depth are essential concepts. You are already familar with graphs and coordinates in $n=1$ and 2, but perhaps $n=3$ is new,



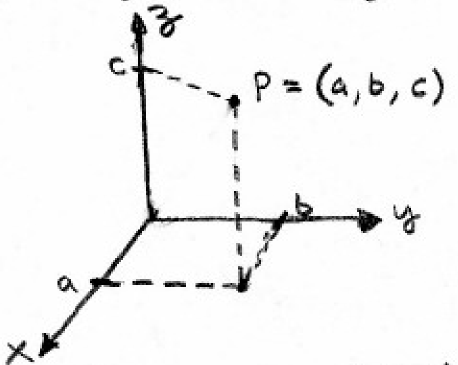
I will almost always use I, but II & III are just a different view of the same. These are all "right handed coordinate systems". We will define that carefully via cross products a little later. There are also left handed coordinate systems, but we will not use them.

Coordinates and Projections

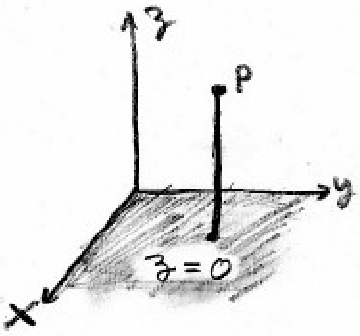
Given a point $P = (a, b, c) \in \mathbb{R}^3$ we define the the

$$P_1 \equiv P_x = a \quad \text{the } x\text{-coordinate of } P$$
$$P_2 \equiv P_y = b \quad \text{the } y\text{-coordinate of } P$$
$$P_3 \equiv P_z = c \quad \text{the } z\text{-coordinate of } P$$

Graphically, assuming $a, b, c > 0$ (P is in the 1st octant)

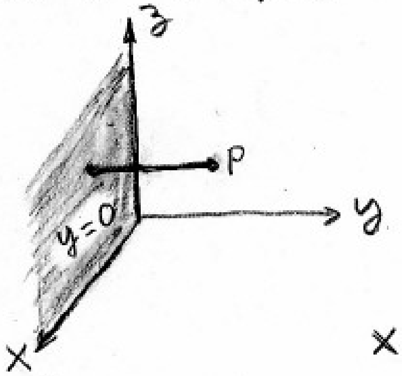


Often it will be useful to employ projections onto the coordinate planes. Let $P = (a, b, c)$ as before,



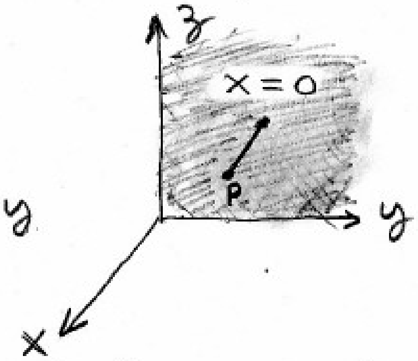
Projection onto the xy-plane

$\pi_{xy}(P) = (a, b, 0)$



Projection onto the xz-plane

$\pi_{xz}(P) = (a, 0, c)$



Projection onto the yz-plane

$\pi_{yz}(P) = (0, b, c)$

Remark: In $\mathbb{R}^2$ the eq^{n}'s $x = 0$ or $y = 0$ would have given us a verticle or horizontal line, but in the context of $\mathbb{R}^3$ they give planes because for each value of z we get a line. If you paste a bunch of lines together they'll make a plane. (provided they're lined up correctly, oh sorry.)

Distance between Points & Line Segments

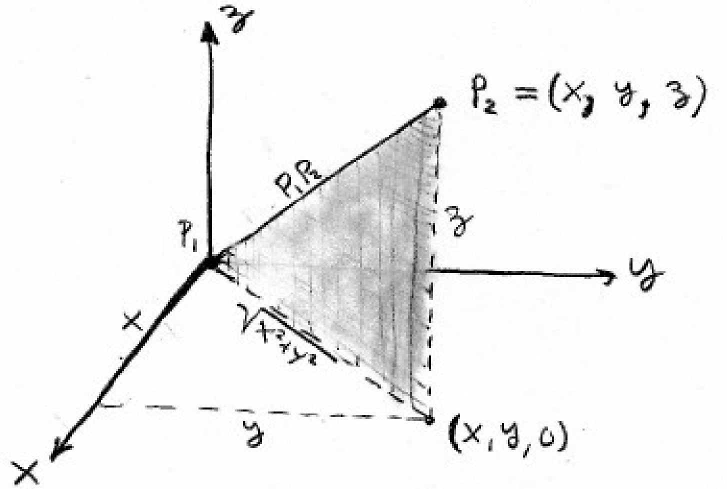
Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ then the directed line segment from P_1 to P_2 is a <u>vector</u>

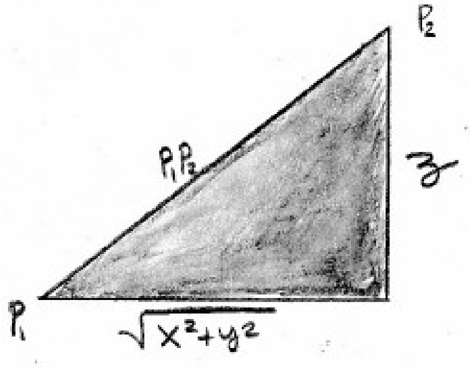
$$P_1P_2 = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1) \equiv P_2 - P_1$$

The distance between P_1 & P_2 is the length of the line segment connecting them. We can show,

$$d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = |P_1P_2|$$

Lets put P_1 at the origin so $P_1 = (0,0,0)$ and $P_2 = (x, y, z)$

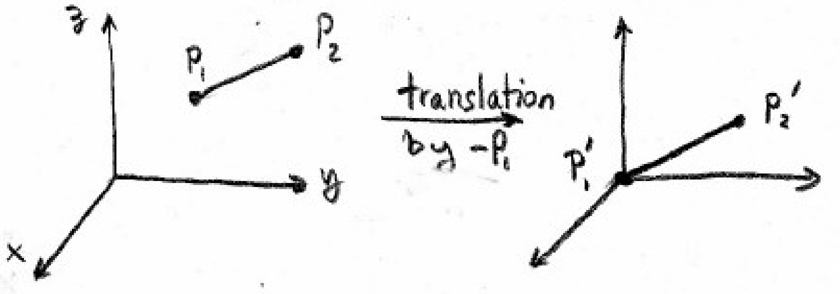




We apply the pythagorean Th$^{\underline{m}}$ to the right triangle in (xy)-plane. Then again apply the pythagorean Th$^{\underline{m}}$ to the shaded Δ,

$$|P_1P_2| = \sqrt{\left(\sqrt{x^2+y^2}\right)^2 + z^2} = \sqrt{x^2+y^2+z^2}.$$

<u>Remark</u>: it is sufficient to prove this for $P_1 = (0,0,0)$ since if $P_1 \neq (0,0,0)$ then we could <u>translate</u> both P_1 and P_2 by $-P_1$ so that $P_1' = (0,0,0)$ and $P_2' = P_2 - P_1$. If we shift both points simultaneously then the same directed line segment connects them (although it's based at the origin instead of P_1)



$T(P) = P - P_1$
translation by $-P_1$ operation.

Length of line segment in n-dimensions

Let $P_1 = (x_i)$ and $P_2 = (y_i)$ for $i = 1, 2, \dots, n$ then

$$|P_1P_2| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

Examples: choose $P_1 = 0$

$n=1 \quad |y| = \sqrt{y^2}$

$n=2 \quad |(y_1, y_2)| = \sqrt{y_1^2 + y_2^2}$

$n=3 \quad |(y_1, y_2, y_3)| = \sqrt{y_1^2 + y_2^2 + y_3^2}$

Remark: this is the Euclidean type of distance. There are other ideas. $\mathbb{R}^n$ equipted with this idea of distance is called Euclidean Space. It is the geometry natural to Classical Newtonian Mechanics.

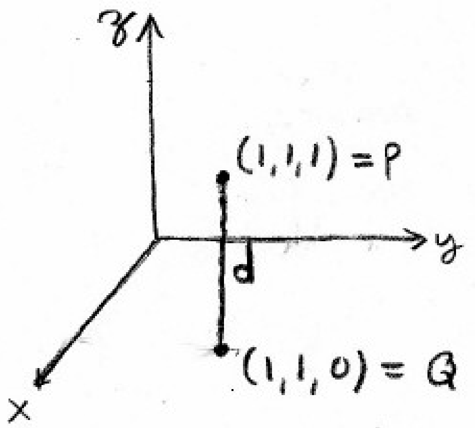
$\boxed{E1}$: THE SPHERE OF RADIUS R based at $(a, b, c) = P$. is defined to be the collection of all $\vec{r} = (x, y, z) \in \mathbb{R}^3$ such that $|\vec{r} - P| = R > 0$. That is

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = R$$

$$\Rightarrow \boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2}$$

sphere of radius R at (a, b, c).

$\boxed{E2}$ Find distance from point $(1,1,1)$ to xy-plane. This is by common agreement the distance to the closest point on the xy-plane to $(1,1,1)$. We can see this must be the distance d in the picture.



$$|P - Q| = |(1,1,1) - (1,1,0)| = |(0,0,1)| = \sqrt{1^2} = \boxed{1}$$

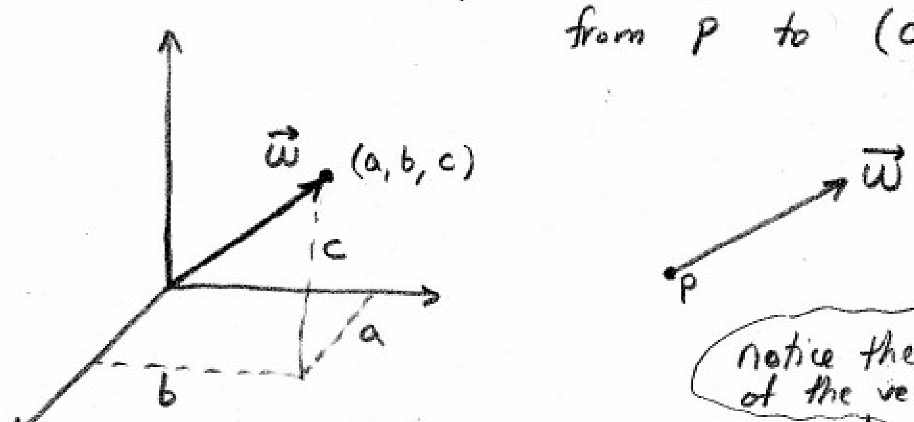
Question: how to do this if we had an arbitrary tilted plane? Its not quite so easy, is it? We will learn tools to help with this.

Vectors in $\mathbb{R}^n$

Abstractly a "vector" can be most anything you can add together. For example, functions, matrices, numbers are all in an abstract sense vectors. The concept of vector for ma 242 is quite speficic though. When we say something is a vector in $\mathbb{R}^n$ then we have in mind a directed line segment. Now we also allow our vectors to move around, that is we identify the following as the same vector,

(Remark: if we wish to distinguish them we can simply affix the qualifier $\vec{v}$ at P_1 or $\vec{v}$ at P_2.)

Since by default $\vec{v}$ at P_1 and $\vec{v}$ at P_2 are identified we can always transport the vector $\vec{v}$ to the origin so that it corresponds uniquely to the point it points to. For example, $\vec{w}$ picture below is transported from P to $(0,0,0)$.



notice the x, y, z - components of the vector are the same as the identified point.

Therefore we identify $\vec{w} = (a,b,c) = \langle a,b,c \rangle$.

Remark: I sometimes add a $\rightarrow$ or $\rightarrow$ on the top of a vector to emphasize its a vector. Some folks demand you always put the $\rightarrow$ on vectors. We will add them when it helps.

Basic Operations on Vectors

Let $V = \langle V_1, V_2, V_3 \rangle$ and $W = \langle W_1, W_2, W_3 \rangle$ be in $\mathbb{R}^3$ and $c \in \mathbb{R}$,

$V + W \equiv \langle V_1 + W_1,\ V_2 + W_2,\ V_3 + W_3 \rangle$	Vector Addition
$cV \equiv \langle cV_1,\ cV_2,\ cV_3 \rangle$	Scalar Multiplication

If we denote the i^{th} component of $V, W \in \mathbb{R}^n$ by V_i or W_i respectively then we can express these rules component wise,

$$(V+W)_i \equiv V_i + W_i$$

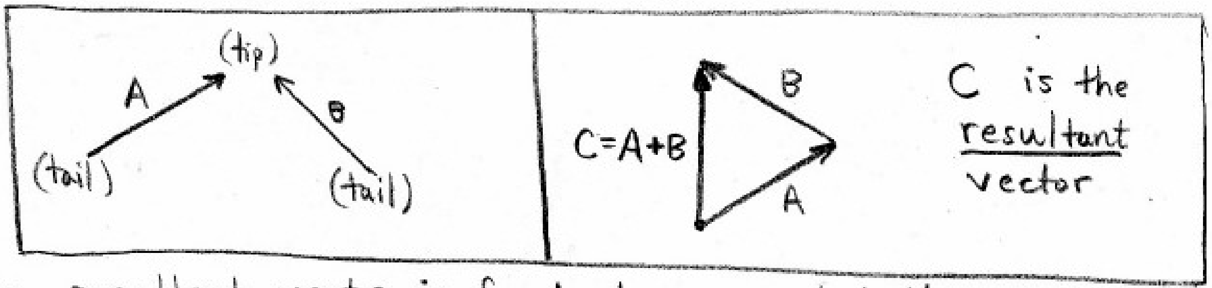
$$(cV)_i \equiv cV_i$$

In other words to add vectors we add the corresponding components. To multiply by a scalar we multiply all the components by the scalar.

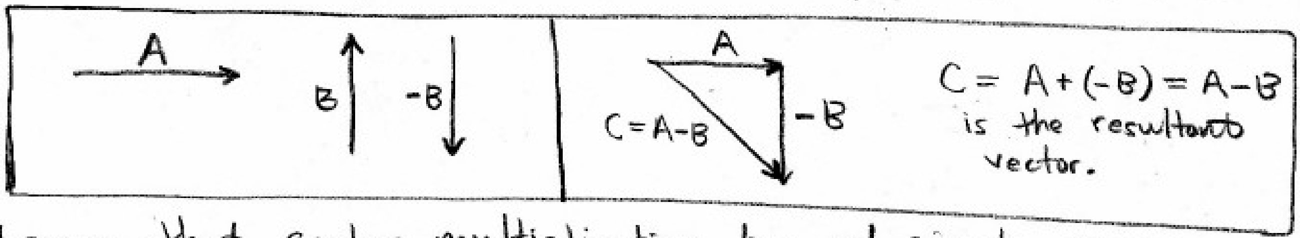
$V = W \iff V_i = W_i \quad \forall i = 1, 2, \ldots, n$	Vector Equality

Geometry of Vector Addition

We add vectors tip to tail,



the resultant vector is found by connected the initial point to the end point which is the tip of B here.



Observe that scalar multiplication by -1 simply reverses the direction of the vector. As usual subtraction is defined to be addition of the additive inverse ($-B$ in the above)

Remark: Pictures are nice and all, but when it comes to details you usually want to break down the vector into its components. Then its easy to work with the components.

Length of Vectors

We define the length of $A \in \mathbb{R}^n$ to be the length of the corresponding line segment. that is $A = \langle A_1, A_2, \dots, A_n \rangle$

$$\boxed{\text{length of } A \equiv |A| \equiv \sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}}$$

(also called the magnitude of A)

Properties: Let $A, B \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then

(i.) $|cA| = |c||A|$ (where $|c|$ denotes absolute value of c)

(ii) $|A| \geq 0$ and $|A| = 0$ iff $A = 0$.

(iii) $|A+B| \leq |A| + |B|$ (triangle inequality)

Proof: we leave (ii) & (iii) for the reader. We prove (i) in case $n = 3$.

$$\begin{aligned} |cA| &= \sqrt{(cA_1)^2 + (cA_2)^2 + (cA_3)^2} \\ &= \sqrt{c^2(A_1^2 + A_2^2 + A_3^2)} \\ &= \sqrt{c^2}\sqrt{A_1^2 + A_2^2 + A_3^2} \\ &= |c||A| \; // . \end{aligned}$$

Unit Vectors

Let $b \in \mathbb{R}^n$ be a nonzero vector then $|b| \neq 0$ and we define the unit vector in the b-direction to be $\hat{b}$.

$$\boxed{\hat{b} \equiv \frac{1}{|b|} b}$$

we can prove that $|\hat{b}| = 1$. Observe,

$$|\hat{b}| = \left| \frac{1}{|b|} b \right| = \frac{1}{|b|}|b| = 1.$$

Remark: Another popular notation to use is $\vec{A}$ = vector then $A = |\vec{A}|$ then we can write that (assume $\vec{A} \neq 0$)

$$\vec{A} = \frac{A}{A}\vec{A} = A\hat{A}$$

this convention says without the "$\longrightarrow$" its the length or magnitude. And we have just shown that any vector can be written as the product of its magnitude (A) and its direction ($\hat{A}$).

Unit vectors in the x, y, z directions

$\hat{i} = \langle 1, 0, 0 \rangle$	$\hat{j} = \langle 0, 1, 0 \rangle$	$\hat{k} = \langle 0, 0, 1 \rangle$

These obviously have length one, they provide a "basis" for $\mathbb{R}^3$. that means they are linearly independent and span $\mathbb{R}^3$. The technical meaning of that need not concern us. Note $(a, b, c) \in \mathbb{R}^3$ then,

$$\begin{aligned} \langle a, b, c \rangle &= \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, c \rangle \\ &= a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle \\ &= a\hat{i} + b\hat{j} + c\hat{k} \end{aligned}$$

We can use either notation. In $\mathbb{R}^n$ the natural basis is $e_1 = \langle 1, 0, \dots, 0 \rangle$, $e_2 = \langle 0, 1, 0, \dots, 0 \rangle, \dots, e_n = \langle 0, 0, \dots, 1 \rangle$.

DOT PRODUCT

Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ then

$$A \cdot B = A_1 B_1 + A_2 B_2 + A_3 B_3 = |A||B| \cos\theta$$

where θ is the angle between A & B. $(0 \leq \theta \leq \pi)$

Def^{n}/ $A \perp B \Leftrightarrow A$ orthogonal to $B \Leftrightarrow A \cdot B = 0 \Leftrightarrow \theta = 90°$

PROPERTIES OF DOT PRODUCT:

Let $A, B, C \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$,

(i.) $A \cdot B = B \cdot A$	Commutative
(ii.) $A \cdot A = \|A\|^2$ a.k.a. $\|A\| = \sqrt{A \cdot A}$	$A \cdot A$ is $(\text{length})^2$
(iii.) $A \cdot (B + C) = A \cdot B + A \cdot C$	distributes over +
(iv.) $A \cdot (\alpha B) = (\alpha A) \cdot B = \alpha (A \cdot B)$	scalars come out
(V.) $e_i \cdot e_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	natural basis is orthogonal
(vi) $A \cdot e_i = A_i$	dot product with e_i selects i^{th} comp.

All of these are straightforward to prove. Usually we deal with $n = 2$ or $n = 3$

PROJECTING THE COMPONENTS WITH $\hat{i}$, $\hat{j}$ and $\hat{k}$

Expanding on (vi) of the PROPERTIES in $n=3$. Let $A=\langle A_1, A_2, A_3\rangle$

$$A\cdot\hat{i} = \langle A_1, A_2, A_3\rangle\cdot\langle 1,0,0\rangle = A_1(1)+A_2(0)+A_3(0) = A_1$$
$$A\cdot\hat{j} = \langle A_1, A_2, A_3\rangle\cdot\langle 0,1,0\rangle = A_1(0)+A_2(1)+A_3(0) = A_2$$
$$A\cdot\hat{k} = \langle A_1, A_2, A_3\rangle\cdot\langle 0,0,1\rangle = A_1(0)+A_2(0)+A_3(1) = A_3$$

Or we could do this in $\hat{i}, \hat{j}, \hat{k}$ notation. The crucial facts are

$\hat{i}\cdot\hat{i}=1$	$\hat{j}\cdot\hat{j}=1$	$\hat{k}\cdot\hat{k}=1$
$\hat{i}\cdot\hat{j}=0$	$\hat{j}\cdot\hat{k}=0$	$\hat{k}\cdot\hat{i}=0$

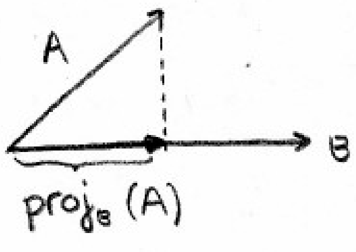
Then I'll redo $A\cdot\hat{i}=A_1$ in the unit-vector notation,

$$\begin{aligned} A\cdot\hat{i} &= (A_1\hat{i}+A_2\hat{j}+A_3\hat{k})\cdot\hat{i} \\ &= A_1\underbrace{\hat{i}\cdot\hat{i}}_{1} + A_2\underbrace{\hat{j}\cdot\hat{i}}_{0} + A_3\underbrace{\hat{k}\cdot\hat{i}}_{0} \quad \text{using Property (iii.)} \\ &= A_1 \end{aligned}$$

<u>Remark</u>: the dot product $A\cdot\hat{i}$ gives us the part of A which is parallel to the $\hat{i}$-direction. This indicates the dot product can be used to measure how closely $\parallel$ to vectors A, B are. We should expect that $A\cdot\hat{B}$ gives the component of A in the $\hat{B}$-direction, and $(A\cdot\hat{B})\hat{B}$ is the part of the vector A which lies in the $\hat{B}$-direction. This motivates the following,

> Scalar Projection of A onto B: $\quad \text{comp}_B(A) \equiv A\cdot\hat{B} = \dfrac{A\cdot B}{|B|}$
>
> Vector Projection of A onto B: $\quad \text{proj}_B(A) \equiv (A\cdot\hat{B})\hat{B} = \dfrac{A\cdot B}{|B|^2}B$

Notice that $|\text{proj}_B(A)| = \left|\frac{A\cdot B}{|B|^2}B\right| = \frac{A\cdot B}{|B|^2}|B| = \frac{A\cdot B}{|B|} = \text{comp}_B(A)$.



- You can show that the other part of A is orthogonal to B. Lets define
$$C = A - \text{proj}_B(A)$$
then $C\perp B$ meaning $B\cdot C=0$.

Can you see where C would be in the picture?

Examples

E3 Given $A = \langle 3, 4\rangle$ and $B = \langle 5, 12\rangle$ find the angle between them and the unit vectors $\hat{A}$ & $\hat{B}$ and the vector/scalar projections,

$$|A| = \sqrt{9+16} = \sqrt{25} = 5 \quad \therefore \boxed{\hat{A} = \tfrac{1}{5}\langle 3,4\rangle}.$$
$$|B| = \sqrt{25+144} = \sqrt{169} = 13 \quad \therefore \boxed{\hat{B} = \tfrac{1}{13}\langle 5,12\rangle}$$

$$A\cdot B = |A||B|\cos\theta = 65\cos\theta \quad \text{\&} \quad A\cdot B = 3(5)+4(12) = 15+48 = 63.$$

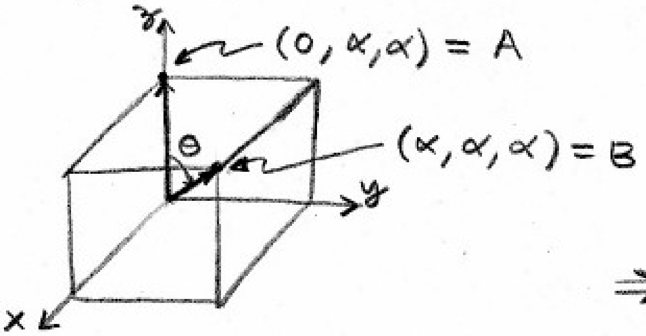
$$\Rightarrow \cos\theta = 63/65 \quad \therefore \boxed{\theta = \cos^{-1}(63/65) = 0.2487 = 14.25^\circ}$$

Then we find

$$\text{comp}_A(B) = B\cdot\hat{A} = \tfrac{1}{5}\langle 5,12\rangle\cdot\langle 3,4\rangle = \tfrac{1}{5}(15+48) = 63/5$$
$$\text{proj}_A(B) = \text{comp}_A(B)\cdot\hat{A} = \tfrac{63}{5}\tfrac{1}{5}\langle 3,4\rangle = \tfrac{63}{25}\langle 3,4\rangle$$

I leave $\text{comp}_B(A)$ and $\text{proj}_B(A)$ for you to figure.

E4 find angle between diagonal of cube and a line on the corner. We place the cube at the origin and let its sides have length α.



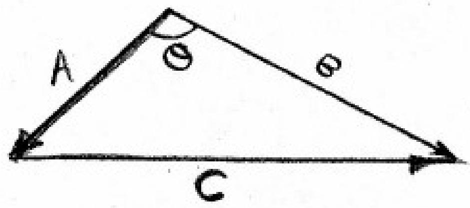
$$|A| = \sqrt{2\alpha^2} = \sqrt{2}\,\alpha$$
$$|B| = \sqrt{3\alpha^2} = \sqrt{3}\,\alpha$$
$$A\cdot B = \langle 0,\alpha,\alpha\rangle\cdot\langle\alpha,\alpha,\alpha\rangle = 2\alpha^2$$
$$\Rightarrow 2\alpha^2 = (\sqrt{2}\alpha)(\sqrt{3}\alpha)\cos\theta$$
$$\therefore \frac{2}{\sqrt{2}\sqrt{3}} = \cos\theta$$
$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = \boxed{65.91^\circ}$$

E5 If a <u>constant</u> force of $F = \langle 1,1,1\rangle$ is applied to a particle that is displaced by $\Delta x = \langle 1,0,0\rangle$ find the work done.

$$W \equiv F\cdot\Delta x = \langle 1,1,1\rangle\cdot\langle 1,0,0\rangle = \boxed{1 = W}$$

<u>Remark:</u> Only the components of the force which are parallel to the displacements do work. Orthogonal forces do no work for example the magnetic force $F = q\,V\times B$ is $\perp$ to the motion.
cross product, our next topic!

QUESTION : how can $A \cdot B = A_1 B_1 + A_2 B_2 + A_3 B_3$ and $A \cdot B = |A||B| \cos\theta$ be the same formula? Lets assume its true as we have been for some pages now, consider the triangle below,



$$C = -A + B$$

(can you see this)

$$\begin{aligned}
|C|^2 = C \cdot C &= (B - A) \cdot (B - A) \\
&= B \cdot B - A \cdot B - B \cdot A + A \cdot A \\
&= |B|^2 + |A|^2 - 2 A \cdot B \\
&= |A|^2 + |B|^2 - 2|A||B| \cos\theta
\end{aligned}$$

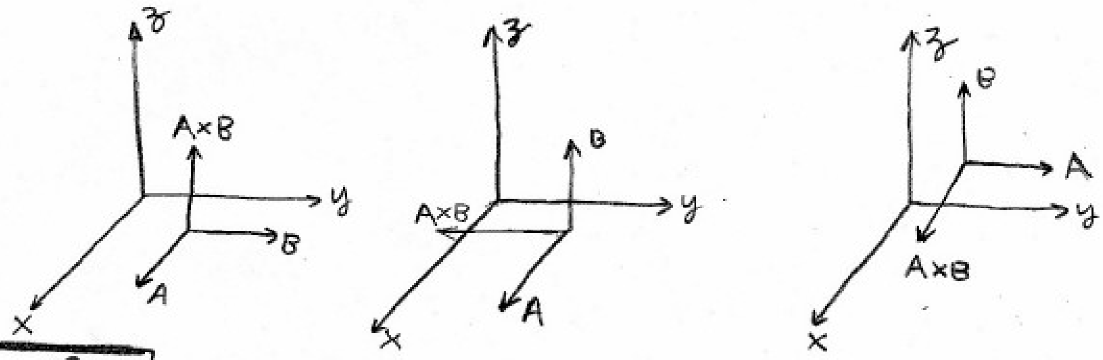
assuming that $A \cdot B = |A||B| \cos\theta$.

But this is the Law of Cosines. So we find that $A \cdot B = |A||B| \cos\theta$ follows from the Law of Cosines. Next we connect to the component formula. First solve for $A \cdot B$,

$$\begin{aligned}
A \cdot B &= \tfrac{1}{2}\left(|A|^2 + |B|^2 - |C|^2\right) \quad , \quad C = B - A \\
&= \tfrac{1}{2}\left(A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - (B_1 - A_1)^2 - (B_2 - A_2)^2 - (B_3 - A_3)^2 \right. \\
&= \tfrac{1}{2}\Big\{ A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - (B_1^2 - 2A_1B_1 + A_1^2) \\
&\qquad - (B_2^2 - 2A_2B_2 + A_2^2) - (B_3^2 - 2A_3B_3 + A_3^2) \Big\} \\
&= A_1 B_1 + A_2 B_2 + A_3 B_3 .
\end{aligned}$$

CROSS PRODUCT

We have seen that the dot product measures how $\parallel$ a pair of vectors is, a natural question to ask is if there is another product which measures how $\perp$ a pair of vectors are? If you think about it a little you'll see this product should output a vector because there are vectors with $\theta = 90°$ yet they are differently $\perp$, for examples



RIGHT HAND RULE

- The vector $A \times B$ points in the direction which is orthogonal to the A & B directions. The right hand rule gives you the direction of $A \times B$. Simply point the fingers of your right hand in the A direction then cross or curl them into the B direction, the orthogonal direction to both A & B where your thumb then points is the $A \times B$ direction.

Def^n/ Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ then

$$A \times B \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \equiv (A_2B_3 - A_3B_2)\hat{i} + (A_3B_1 - A_1B_3)\hat{j} + (A_1B_2 - A_2B_1)\hat{k}$$

$$A \times B \equiv \langle A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1 \rangle$$

Or more geometrically $A \times B = |A||B| \sin\theta\, \hat{A} \times \hat{B}$ where the vector $\hat{A} \times \hat{B}$ is given by the right hand rule.

- these definitions are in fact equivalent. I delay the proof till we discussed the cross product further.

Def^n/ A coordinate system (x_1, x_2, x_3) is RIGHT HANDED iff its coordinate vectors $\hat{x}_1, \hat{x}_2, \hat{x}_3$ satisfy $\hat{x}_1 \times \hat{x}_2 = \hat{x}_3$

PROPERTIES OF CROSS PRODUCT : Let $A,B,C \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$	
(i.) $A \times B = -B \times A$	skew symmetric
(ii) $(\alpha A)\times B = A \times (\alpha B) = \alpha (A \times B)$	scalars pull out
(iii) $A \times (B+C) = A\times B + A \times C$	distributes over +
(iv.) $A\times(B\times C) + B\times(C\times A) + C\times(A\times B) = 0$	JACOBI IDENTITY measures nonassociativity
(v.) $A \cdot (B \times C) = (A\times B)\cdot C$	
(vi.) $A\times(B\times C) = (A\cdot C)B - (A\cdot B)C$	
(vii) $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$	$e_m \times e_n = \epsilon_{mnk} e_k$.

Proof: (i), (ii), (iii), (vii) follow quickly from the definition plus the fact that vector addition is linear. The more subtle parts (iv), (v) and (vi.) I have proved in the homework. In (vii.) I comment $e_m \times e_n = \epsilon_{mnk} e_k$. The meaning of this is expanded on in the hwk. sol^{n}. Also you can look at my ma 430 notes for more details, or ask me about the repeated index notation (not a req^{d} topic.) //.

Examples

E6 Let $A = \langle 1,2,3 \rangle$, $B = \langle 4,5,6\rangle$ find $A\times B$;

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

← notation for 3×3 determinant, technically this is a little bogus since all the entries are supposed to be of the same type. It works though.

$$= \hat{i}\begin{vmatrix} 2 & 3 \\ 5 & 6\end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 4 & 6\end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 4 & 5\end{vmatrix}$$

← Laplace expansion into subdeterminants, its a definition if you like.

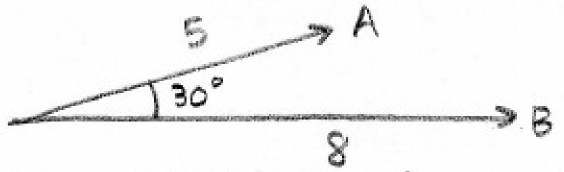
$$= \hat{i}\big(2(6)-3(5)\big) - \hat{j}\big(1(6)-3(4)\big) + \hat{k}\big(1(5)-2(4)\big)$$

$$= -3\hat{i} + 6\hat{j} - 3\hat{k}$$

$$= \boxed{\langle -3, 6, -3\rangle = A\times B}$$

Notice $A\cdot(A\times B) = \langle 1,2,3\rangle \cdot \langle -3,6,-3\rangle = -3+12-9 = 0$
and $B\cdot(A\times B) = \langle 4,5,6\rangle\cdot\langle -3,6,-3\rangle = -12+30-18 = 0$.
This proves $A, B \perp A\times B$. This is a good check to do on your $A\times B$ calculation, dot products are easy.

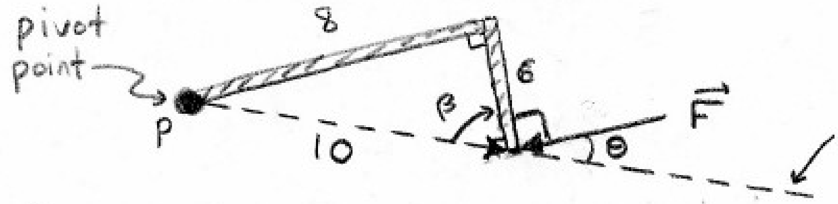
E7 Consider the vectors pictured below. Find $A \times B$



$$|A\times B| = |A||B|\sin\theta = 40\sin 30^\circ = 20.$$

By the right hand rule $A \times B = 20\hat{n}$ where $\hat{n}$ points into page.

E8 The torque τ about a radial arm $\vec{r}$ created by a force $\vec{F}$ is defined to be $\vec{\tau} = \vec{r} \times \vec{F}$. Calculate the torque generated by $\vec{F}$ in the picture below,



radial line to where $\vec{F}$ is applied, Note $|\vec{r}| = \sqrt{64+36} = 10.$

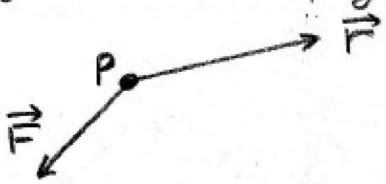
Notice that $\theta + 90^\circ + \beta = 180^\circ$. Then notice $\tan\beta = 8/6$ thus $\beta = \tan^{-1}(8/6) = 53.13^\circ$. Calculate then

$$\theta = 90^\circ - \beta = 36.87^\circ$$

we see that the acute angle at P is in fact θ so $\sin\theta = \frac{6}{10}$.

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta = 10F\frac{6}{10} = \boxed{6F = \tau}$$

and $\vec{\tau}$ is into the page by right hand rule applied to.

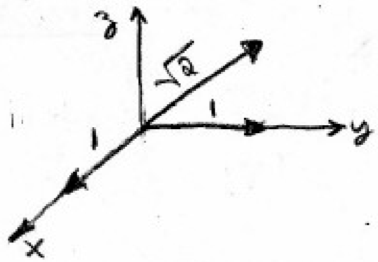


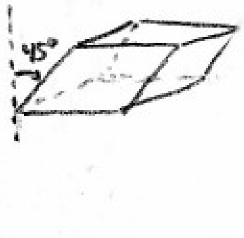
this means this force will produce a motion like ↻.

E9 The volume of a parallelepiped with edges A, B and C out of a common vertex is given by

$$V = |A\cdot(B\times C)|$$

For example let $A = \langle 0,1,1\rangle$, $B = \langle 1,0,0\rangle$, $C = \langle 0,1,0\rangle$ then $B\times C = \hat{\imath}\times\hat{\jmath} = \hat{k}$ and $A\cdot(B\times C) = 1$. Thus





has volume $\boxed{V = 1}$

Lagrange's Identity

Use number 32 of §9.5 with $a=c$ and $b=d$ to obtain

$$(a\times b)\cdot(a\times b) = \begin{vmatrix} a\cdot a & b\cdot a \\ a\cdot b & b\cdot b \end{vmatrix}$$

thus,

$$\begin{aligned} |a\times b|^2 &= |a|^2|b|^2 - (a\cdot b)^2 \\ &= |a|^2|b|^2 - (|a||b|\cos\theta)^2 \\ &= |a|^2|b|^2(1-\cos^2\theta) \\ &= |a|^2|b|^2\sin^2\theta \quad \Rightarrow \quad |a\times b| = |a||b||\sin\theta| \end{aligned}$$

$$\Rightarrow \quad |a\times b| = |a||b|\sin\theta$$

(Since we assume $0\leq\theta\leq\pi$ where $\sin\theta \geq 0$.)

This shows that the component formula for $A\times B$ is consistent with $A\times B = |A||B|\sin\theta\,\hat{n}$ at least in terms of magnitudes. The directional compatibility follows from case wise examination of $\hat{i}, \hat{j}, \hat{k}$'s cross products.

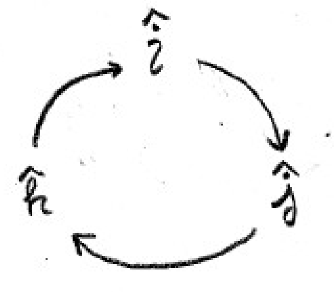
$$\hat{i}\times\hat{j} = \hat{k} \qquad \text{(by right hand rule.)}$$

Verses,

$$\hat{i}\times\hat{j} = \langle 1,0,0\rangle \times \langle 0,1,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{k}(1-0) = \hat{k}.$$

Clearly the geometric and algebraic formulas give same result. Then we could check the same for the other unit vectors.

Remark: I find the following pictures helpful to recall the basic cross products.



$\Rightarrow$

$$\left.\begin{aligned} \hat{i}\times\hat{j} &= \hat{k} \\ \hat{j}\times\hat{k} &= \hat{i} \\ \hat{k}\times\hat{i} &= \hat{j} \end{aligned}\right\} \text{follow the arrow}$$

$$\left.\begin{aligned} \hat{j}\times\hat{i} &= -\hat{k} \\ \hat{k}\times\hat{j} &= -\hat{i} \\ \hat{i}\times\hat{k} &= -\hat{j} \end{aligned}\right\} \text{goes opposite the arrow.}$$

$$\xrightarrow{\hat{i}\ \ \hat{j}\ \ \hat{k}\ \ \hat{i}}$$

Same idea, follow arrow to get $\hat{j}\times\hat{k} = \hat{i}$ etc... against arrow $\hat{i}\times\hat{k} = -\hat{j}$