We switch gears a little and return to study parametrized curves like $r(t)$. Notice $r: \mathbb{R} \longrightarrow \mathbb{R}^3$ it is the assignment

$$t \longmapsto r(t) = \langle x(t), y(t), z(t) \rangle$$

the key here is that the domain is $\mathbb{R}$ (or generally some subset) so we can do all the usual calculus; limits, derivatives & integrals. We simply do them __componentwise__. Notice $r(t)$ is actually a vector of three functions from $\mathbb{R} \longrightarrow \mathbb{R}$, these functions $x(t), y(t), z(t)$ are the __component functions__.

Def^n/ Let $r(t) = \langle x(t), y(t), z(t) \rangle$ then

$$\lim_{t \to a} r(t) = \langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \rangle$$

$$\frac{dr}{dt} \equiv r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\int_a^b r(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$$

Likewise for one sided limits and indefinite integrals.

__Remark__: ok, so I made Th^m (2) the def^n because I can. The geometric interpretation of $r'(t)$ as the slope of the tangent line at $r(t)$ needs a little generalization, afterall $r'(t)$ is a vector.

$$\boxed{\ell(t) = r_0 + t\,r'(t_0)}$$ eq^n of tangent line.

$\uparrow$ gives the direction vector to the curve $r(t)$ at the point $r(t_0) = r_0$

as before the tangent line is seen as limiting case of secant lines approaching r_0

$$\frac{dr}{dt} = \lim_{h \to 0} \left(\underbrace{\frac{r(t+h) - r(t)}{h}}_{\substack{\text{direction vector} \\ \text{of secant line.}}} \right)$$
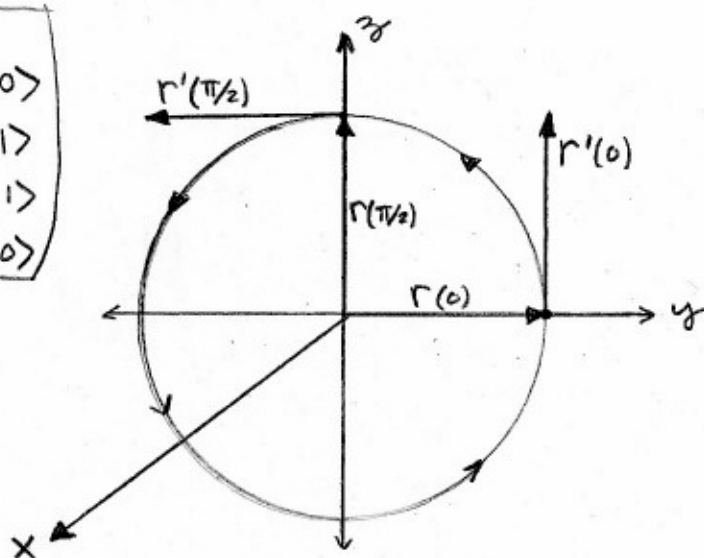
E26 Consider $r(t) = \langle 0, \cos t, \sin t \rangle$. Find $r'(t)$ and plot $r'(0)$ and $r'(\pi/2)$. Then calculate $\int r(t)dt$.

$$r'(t) = \frac{d}{dt}\left(\langle 0, \cos t, \sin t \rangle\right)$$

$$= \langle \frac{d}{dt}(0), \frac{d}{dt}(\cos t), \frac{d}{dt}(\sin t) \rangle$$

$$= \boxed{\langle 0, -\sin t, \cos t \rangle = \frac{dr}{dt}}$$

$$\int r(t)dt = \langle \int 0 \, dt, \int \cos t \, dt, \int \sin t \, dt \rangle$$

$$= \langle C_1, \sin t + C_2, -\cos t + C_3 \rangle$$

$$= \boxed{\langle 0, \sin t, -\cos t \rangle + C} = \int r(t)dt$$

Evaluate:
$r(0) = \langle 0, 1, 0 \rangle$
$r'(0) = \langle 0, 0, 1 \rangle$
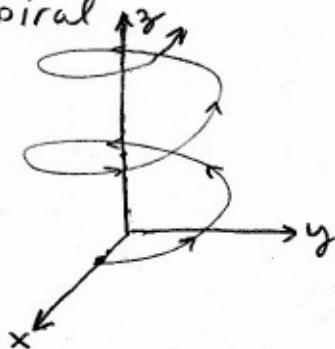$r(\pi/2) = \langle 0, 0, 1 \rangle$
$r'(\pi/2) = \langle 0, -1, 0 \rangle$

We see that $r(t)$ is a circle of radius one in the yz-plane.

Algebraically we can prove this,
$x = 0$
$y = \cos t$
$z = \sin t$
$\Rightarrow y^2 + z^2 = \sin^2 t + \cos^2 t = 1$

E27 Consider $r(t) = \langle \cos t, \sin t, t \rangle$. Plot and find $r'(t)$. This projects to the circle $\cos^2 t + \sin^2 t = x^2 + y^2 = 1$ in the xy-plane, its a spiral

$$r'(t) = \langle -\sin t, \cos t, 1 \rangle$$

Notice $\frac{dz}{dt} = 1$ so this spiral rises with slope one for all time.

Remark: to give a complete description we should give $r(t)$ as well as the allowed values for t. This becomes more important as we go on.

E28) Study $r(t) = r_0 + tv$ where r_0, v are fixed vectors independent of time. Let $v = \langle a, b, c \rangle$,

$$r'(t) = \frac{d}{dt}(r_0 + tv) = \frac{dr_0}{dt} + \frac{d}{dt}\langle ta, tb, tc \rangle = \langle a, b, c \rangle = v.$$

the tangent vector to this curve is $r'(t) = v$ for all time. We find that a line in $\mathbb{R}^3$ has constant direction vectors.

PROPERTIES OF $\frac{d}{dt}$ AND $\int$ on VECTOR VALUED FUNCTIONS OF $\mathbb{R}$

Let A and B be vector valued functions of $\mathbb{R}$,

(i.) $\frac{d}{dt}[A + B] = \frac{dA}{dt} + \frac{dB}{dt}$

(ii.) $\frac{d}{dt}[cA] = c\frac{dA}{dt}$

(iii) $\frac{d}{dt}[fA] = \frac{df}{dt}A + f\frac{dA}{dt}$: where f is a real-valued function of t.

(iv.) $\frac{d}{dt}[A \cdot B] = A'(t) \cdot B(t) + A(t) \cdot B'(t)$

(v.) $\frac{d}{dt}[A \times B] = \frac{dA}{dt} \times B + A \times \frac{dB}{dt}$

(vi.) $\frac{d}{dt}[A(f(t))] = \frac{dA}{dt}(f(t))\frac{df}{dt} = A'(f(t))f'(t)$: Chain Rule

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Proof: these would all be reasonable test questions to prove. I'll do several, they all follow from the ordinary linearity of $\frac{d}{dt}$, product rule or in the case of (vi.) the chain rule. Denote $A(t) = \langle A_1(t), A_2(t), A_3(t) \rangle$ and $B(t) = \langle B_1(t), B_2(t), B_3(t) \rangle$. I begin with (ii).

$$\frac{d}{dt}(cA) = \frac{d}{dt}\langle cA_1, cA_2, cA_3 \rangle$$

$$= \langle \frac{d}{dt}(cA_1), \frac{d}{dt}(cA_2), \frac{d}{dt}(cA_3) \rangle$$

$$= \langle c\frac{dA_1}{dt}, c\frac{dA_2}{dt}, c\frac{dA_3}{dt} \rangle$$

$$= c\langle A_1'(t), A_2'(t), A_3'(t) \rangle = c\frac{dA}{dt} \; /\!/.$$

Proof: these are more fun if you know Einstein's repeated index notation. In short in that notation we write $A = A_i e_i$, $B = B_j e_j$, $A \cdot B = A_i B_i$ and of course $A \times B = \epsilon_{ijk} A_i B_j e_k$ where the $\{e_i\}_{i=1}^{3}$ is the standard basis of $\mathbb{R}^3$, $e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

and $e_i \times e_j = \epsilon_{ijk} e_k$, here ϵ_{ijk} is the antisymmetric symbol.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k = (1,2,3) \text{ or } (2,3,1), (3,1,2) \\ -1 & \text{if } (3,2,1), (2,1,3), (1,3,2) \\ 0 & \text{if any } i,j,k \text{ repeats} \end{cases}$$

Having concluded our crash course in index trickery (not req'd topic for ma 242!)

(iii.) $\dfrac{d}{dt}[fA] = \dfrac{d}{dt}[fA_i e_i]$ ⎤
$\qquad = \dfrac{d}{dt}[fA_i] e_i$ ⎦ this is our def^n.

$\qquad = \left(\dfrac{df}{dt} A_i + f \dfrac{dA_i}{dt}\right) e_i$: plain-old product rule applied thrice.

$\qquad = \dfrac{df}{dt} A + f \dfrac{dA}{dt}$.

(iv.) $\dfrac{d}{dt}[A \cdot B] = \dfrac{d}{dt}[A_i B_i]$

$\qquad = \dfrac{dA_i}{dt} B_i + A_i \dfrac{dB_i}{dt}$: again the product rule three times once for each value of i.

$\qquad = \dfrac{dA}{dt} \cdot B + A \cdot \dfrac{dB}{dt}$.

(v.) $\dfrac{d}{dt}[A \times B] = \dfrac{d}{dt}\left[\epsilon_{ijk} A_i B_j e_k\right]$

$\qquad = \left(\dfrac{d}{dt} \epsilon_{ijk} A_i B_j\right) e_k$

$\qquad = \left(\epsilon_{ijk} \dfrac{d}{dt}[A_i B_j]\right) e_k$ ⎤ ϵ_{ijk} is constant with respect to t.

$\qquad = \epsilon_{ijk}\left(\dfrac{dA_i}{dt} B_j + A_i \dfrac{dB_j}{dt}\right) e_k$ ⎦

$\qquad = \epsilon_{ijk} \dfrac{dA_i}{dt} B_j e_k + \epsilon_{ijk} A_i \dfrac{dB_j}{dt} e_k = \dfrac{dA}{dt} \times B + A \times \dfrac{dB}{dt}$ ⫽

Proof continued: You can prove these by brute-force, that is just write out all the terms. I like the Einstein notation because it allows us to focus on the crucial point which is that all these properties are simply stolen from calculus I due to the linearity of the vector construction, dot-product and cross product.

(vi)
$$\frac{d}{dt}\left(A(f(t))\right) = \frac{d}{dt}\left[A_j(f(t))\, e_j\right]$$

$$= \left(\frac{d}{dt}\left[A_j(f(t))\right]\right)e_j$$

$$= \left(\frac{dA_j}{dt}\Big|_{f(t)}\frac{df}{dt}\right)e_j$$

$$= \left(\frac{df}{dt}\right)\left(A_j'(f(t))\, e_j\right)$$

$$= \frac{df}{dt}\, A'(f(t))$$

$$= A'(f(t))\, f'(t) = \frac{dA}{dt}\Big|_{f(t)}\frac{df}{dt} \quad /\!/.$$

• many unecessary steps here. but do you know all these notations?

E29 Suppose $F(t) = \langle t^3, t^2, t\rangle$ and $g(t) = \underline{\cosh(t)}$.

$$F'(t) = \langle 3t^2, 2t, 1\rangle$$

$$\cosh(t) \equiv \tfrac{1}{2}\left(e^t + e^{-t}\right)$$
$$\sinh(t) \equiv \tfrac{1}{2}\left(e^t - e^{-t}\right)$$

$$\frac{d}{dt}\left[F(g(t))\right] = F'(g(t))\cdot g'(t)$$

$$= \langle 3\cosh^2(t), 2\cosh(t), 1\rangle \cdot \sinh(t)$$

$$= \langle 3\sinh(t)\cosh^2(t), 2\sinh(t)\cosh(t), \sinh(t)\rangle$$

Of course you get the same answer if you first compose then differentiate.

$$F(g(t)) = \langle \cosh^3 t, \cosh^2 t, \cosh t\rangle$$

$$\Rightarrow \frac{d}{dt}\left(F(g(t))\right) = \langle 3\cosh^2(t)\sinh t, 2\cosh t\sinh t, \sinh t\rangle$$

However, a sufficiently diabolical test maker can easy make the 2$^{\text{nd}}$ approach inaccessible.

Given a curve parametrized by $r(t)$ then the tangent line to the curve at $r(t_0)$ is parametrized by the line $\ell(t)$

$$\ell(t) = r(t_0) + t\,r'(t_0)$$

[E30] Consider the curve $r(t) = \langle \cos t, \sin t, \frac{1}{100}\sin(100t)\rangle$, $0 \le t \le 2\pi$, plot the curve and the tangent line at $r(\pi/4)$. Also find the eq^n of the tangent line.

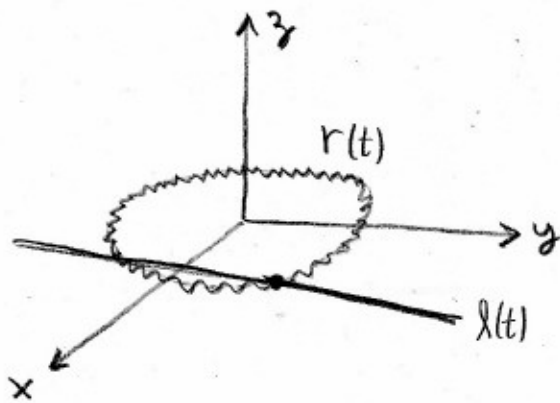$$r'(t) = \langle -\sin t, \cos t, \frac{100}{100}\cos(100t)\rangle$$

$$r(\pi/4) = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle$$

$$r'(\pi/4) = \langle -1/\sqrt{2}, 1/\sqrt{2}, \cos(25\pi)\rangle = \langle -1/\sqrt{2}, 1/\sqrt{2}, -1\rangle$$

Thus the eq^n of the tangent line to $r(\pi/4)$ is

$$\ell(t) = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle + t\langle -1/\sqrt{2}, 1/\sqrt{2}, -1\rangle$$
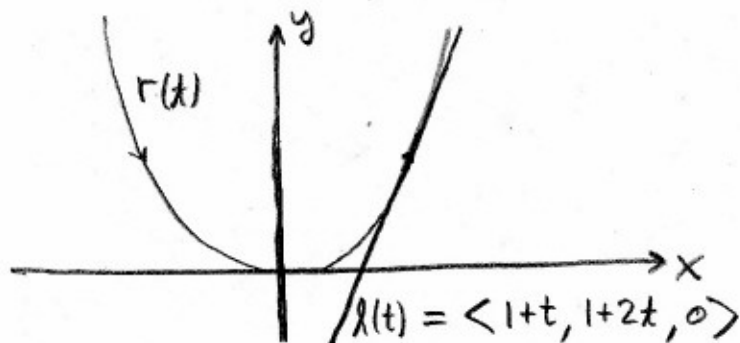
Notice $\frac{-1}{100} \le z \le \frac{1}{100}$ so $\approx z = 0$ and the curve is simply $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Roughly,

Remark: conceptually the "t" in $r(t)$ and $\ell(t)$ is distinct. Probably it'd be better to use a different parameter for ℓ.

[E31] find tangent line to $r(t) = \langle t, t^2, 0\rangle$ at $t=1$.
Note $r'(t) = \langle 1, 2t, 0\rangle$ thus $\boxed{\ell(t) = \langle 1,1,0\rangle + t\langle 1,2,0\rangle}$

$$\ell(t) = \langle 1+t, 1+2t, 0\rangle$$

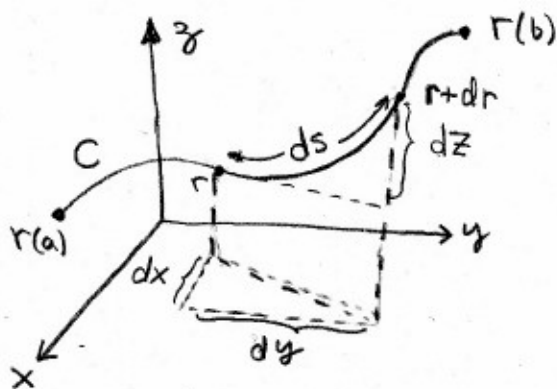the curve and tangent line both lie in $z=0$ so this is easier to picture.

We study a path $r : [a, b] \longrightarrow \mathbb{R}^3$. The point set $r[a, b] \subset \mathbb{R}^3$ is called a curve. There are many paths corresponding to the same curve, however much about the curve can be calculated from an arbitrary path. These properties of the curve are called intrinsic. Essentially the curvature ($\mathbb{K}$), torsion (τ) and arclength (s) characterize curves upto a translation (I'll give careful Th^m later). The curvature and torsion arise in the study of the moving frame T, N, B as we shall see. To begin we study arclength.

ARCLENGTH

Imagine a curve C in $\mathbb{R}^3$, place a length ds of string along the curve then by the distance formula,

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

where dx, dy, dz are the displacements in x, y, z necessary to reach from r to $r + dr$ where $r + dr$ is the place on the curve reach by going ds from r.

Now parametrize the curve by $r : [a, b] \longrightarrow C \subset \mathbb{R}^3$. We have that $dx = \frac{dx}{dt} dt$, $dy = \frac{dy}{dt} dt$, $dz = \frac{dz}{dt} dt$ for the infinitesimal arc considered, dt = time to traverse ds along C.

$$ds = \sqrt{\left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2 + \left(\frac{dz}{dt}dt\right)^2} \Rightarrow ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \; dt$$

$$\therefore \boxed{S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \; dt}$$

arclength of C

Remark: technically we mean to approx. the path by piecewise polygonal path then take limiting process. The infinitesimals are simply an abbreviation for that.

The path $r: [a, b] \to C \subset \mathbb{R}^3$ with parameter t has

$$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$|r'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Thus the arclength can be written as for C from $r(a)$ to $r(b)$,

$$\boxed{S = \int_a^b |r'(t)| \, dt}$$

[E32] Consider $r(t) = \langle R \cos t, R \sin t \rangle$ for $R > 0$. Notice that $r'(t) = \langle -R \sin t, R \cos t \rangle$ thus $|r'(t)| = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R$. We want a circle so $0 \le t \le 2\pi$ then $a = 0$, $b = 2\pi$ so

$$S = \int_0^{2\pi} R \, dt = Rt \Big|_0^{2\pi} = \boxed{2\pi R}$$

the arclength of a circle is $2\pi R$, otherwise we'd be in trouble!

[E33] A helix with slope b is given by $r(t) = \langle a \cos t, a \sin t, bt \rangle$ for $0 \le t \le 2\pi$ (could let t keep going if you want the helix to continue on ward.) anyway calculate, we assume a, b are constants.

$$r'(t) = \langle -a \sin t, a \cos t, b \rangle$$

$$|r'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\therefore S = \int_0^{2\pi} \sqrt{a^2 + b^2} \, dt = \left(\sqrt{a^2 + b^2} \right) t \Big|_0^{2\pi} = \boxed{2\pi \sqrt{a^2 + b^2}}$$

Of course when $b = 0$ we get a circle and we recover $2\pi a$ in that case (a good check of things here.)

Remark: to calculate arclength of C we need the curve to be smooth, well at least once differentiable. If the curve has finitely many kinks in it then we can find the total length by adding the differentiable segments lengths together.

Suppose we have a curve C with a nonstop parametrization $r : [a, b] \longrightarrow C \subset \mathbb{R}^3$. (nonstop means $r'(t) \neq 0 \quad \forall t \in [a, b]$ also we can allow $b = \infty$) then the arclength from $r(a)$ upto $r(t)$ is itself a function of t given by

$$s(t) = \int_a^t |r'(\tau)| \, d\tau$$

τ = dummy variable of integration could use $t', \bar{t}, u$ @ whatever but not t.

Consider then, by the fundamental Th of calculus,

$$(*) \quad \frac{ds}{dt} = \frac{d}{dt} \int_a^t |r'(\tau)| \, d\tau = |r'(t)| \equiv \text{Speed of } r(t) \text{ at time } t.$$

By assumption $|r'(t)| \neq 0 \quad \forall t \Rightarrow \frac{ds}{dt} > 0 \quad \forall t$

thus s is a strictly increasing function of $t \Rightarrow s(t)$ can be inverted to give $t(s)$. Thus we can then write $r(s) \equiv r(t(s))$, replace t with t in terms of s.

E34 The helix $r(t) = \langle a \cos t, a \sin(t), bt \rangle \qquad 0 \le t \le 2\pi$

we found $|r'(t)| = \sqrt{a^2 + b^2}$ thus,

$$s(t) = \int_0^t \sqrt{a^2 + b^2} \, d\tau = (\sqrt{a^2 + b^2}) \tau \Big|_0^t = t\sqrt{a^2 + b^2}$$

$$\Rightarrow t = \frac{s(t)}{\sqrt{a^2 + b^2}} \quad \text{or changing notation, } t(s) = \frac{s}{\sqrt{a^2 + b^2}}$$

- it may be wise to suppress the of (t) and of (s) dependence here. Be careful not to confuse $s(t)$ with s times t. We find that

$$r(t(s)) = \boxed{r(s) = \left\langle a \cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right), a \sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right), b\frac{s}{\sqrt{a^2 + b^2}} \right\rangle}$$

E 35 Let $r(t) = \langle t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3 \rangle$, $0 \leq t \leq 1$. Find the arclength function.

$$r'(t) = \langle 1, \sqrt{2}t, t^2 \rangle \implies |r'(t)| = \sqrt{1 + 2t^2 + t^4} = \sqrt{(1+t^2)^2}$$

Thus,

$$s = \int_0^t \sqrt{(1+\tau^2)^2}\, d\tau = \int_0^t (1+\tau^2)\, d\tau = \left(\tau + \tfrac{1}{3}\tau^3\right)\Big|_0^t = \boxed{t + \tfrac{1}{3}t^3 = s}$$

I would like to solve this for t, however it seems out of reach, at least in terms of elementary functions. <u>Bonus point</u> if you can find for me $f(s)$ which is a power series for which $f(s) + \frac{1}{3}(f(s))^3 = s$ that is a power series sol^a for t in terms of just s.

<u>Remark</u>: $s(t)$ maybe impossible to calculate in a nice closed form and even when it is $t(s)$ may be very subtle to calculate. Even so the arclength plays an important allbeit implicit role.

Proposition: A path $r: [0, b] \to C$ which has arclength as its parameter has unit-speed $\forall s$.

<u>Proof</u>: unit speed means $|r'(s)| = 1$. Consider the chain-rule,

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = \frac{dr}{ds}|r'(t)| \qquad \text{using } (*) \text{ on } 271$$

$$\implies \frac{dr}{ds} = \frac{1}{|r'(t)|}\frac{dr}{dt} = \frac{r'(t)}{|r'(t)|} \qquad \left(\begin{array}{l}\text{recall we assume} \\ r(t) \text{ is nonstop.}\end{array}\right)$$

$$\implies \left|\frac{dr}{ds}\right| = \left|\frac{r'(t)}{|r'(t)|}\right| = \frac{|r'(t)|}{|r'(t)|} = 1 \quad //.$$

<u>Remark</u>: the "speed" of a curve is the change in arclength with respect to the change in parameter so the change in arclength w.r.t. arclength is one.

to begin we define the unit tangent vector to be the tangent vector normalized to length one. Assume $r : [a,b] \to C \subset \mathbb{R}^3$ is a non stop ($r'(t) \neq 0 \quad \forall t \in [a,b]$) path

$$\boxed{\text{Def}^n / \quad T \equiv \frac{r'(t)}{|r'(t)|} = \frac{1}{|dr/dt|} \frac{dr}{dt} = \frac{dr}{ds} \quad \text{for arclength } s}$$

$\boxed{E\,36}$ $r(t) = \langle a \cos t, a \sin t, bt \rangle, \quad 0 \leq t \leq 2\pi$: the helix,

$$T(t) = \frac{r'(t)}{|r'(t)|} = \boxed{\frac{1}{\sqrt{a^2+b^2}} \langle -a \sin t, a \cos t, b \rangle = T(t)}$$

Using $|r'(t)| = \sqrt{a^2+b^2}$ as we found previously in $\boxed{E\,33}$, alternatively recall from $\boxed{E\,34}$ that we found C parametrized by arclength,

$$r(s) = \left\langle a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right\rangle$$

$$T(s) = \frac{dr}{ds} = \underline{\frac{1}{\sqrt{a^2+b^2}} \left\langle -a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), b \right\rangle} \quad \longleftarrow \boxed{\begin{array}{l} \text{I factored} \\ \text{out the} \\ \frac{1}{\sqrt{a^2+b^2}} \\ \text{from the} \\ \text{chain rules.} \end{array}}$$

consistent with $T(t)$ found above since from $\boxed{E\,34}$ $\quad t = \frac{s}{\sqrt{a^2+b^2}}$.

$$\boxed{\begin{array}{l} \text{PROPOSITION: Assume that } r : [a,b] \to C \subset \mathbb{R}^3 \text{ is non stop then} \\[4pt] \text{(i.)} \quad T'(t) \text{ is orthogonal to } T(t) \quad \forall t \in [a,b]. \\[4pt] \text{(ii.)} \quad \left.\dfrac{dT}{dt}\right|_{t=t_0} = \text{angular rate of change with respect to } t \\ \qquad \text{of the direction of } T \text{ when } t = t_0. \end{array}}$$
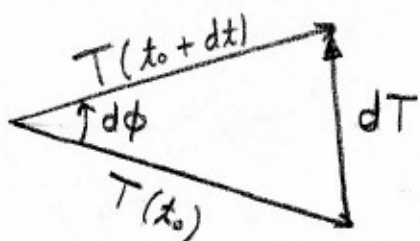
Proof: the proof of (i.) is easy, (ii) requires a little thought.

(i.) $\quad T \cdot T = |T|^2 = 1 \quad$ then differentiate and use property
(iv) from ⟨265⟩ to obtain,

$$\frac{d}{dt}(1) = \frac{dT}{dt} \cdot T + T \cdot \frac{dT}{dt} \Rightarrow 0 = 2T \cdot \frac{dT}{dt} \quad \therefore \quad \underline{T \cdot T' = 0} \quad //.$$

Proof continued:

For (ii.) consider, the infinitesimal argument below,

$$\frac{d\phi}{dt}\Big|_{t=t_0} \stackrel{?}{=} \left|\frac{dT}{dt}\right|_{t=t_0}$$

this is what we seek to show.

Notice that the Law of Cosines yields, use $|T|=1$ to simplify,

$$|dT|^2 = |T(t_0+dt)|^2 + |T(t_0)|^2 - 2|T(t_0+dt)||T(t_0)|\cos(d\phi)$$

$$|dT|^2 = 2 - 2\cos(d\phi) = 2 - 2(1 - \tfrac{1}{2}(d\phi)^2 + \cdots)$$

$$|dT|^2 = (d\phi)^2 \Rightarrow |dT| = d\phi \Rightarrow \left|\frac{dT}{dt}\right|_{t=t_0} = \frac{d\phi}{dt}\Big|_{t=t_0}. \; /\!/.$$

Remark: See Colley's Text for a more conventional proof if you wish.

> Def^n/ The curvature $\mathbb{K}$ of a path $r: [a,b] \to C \subset \mathbb{R}^3$ is the angular rate of change of the direction of T per unit change in the distance along the path.

this def^n is quite geometric, we use the Prop. on ②⑦③ plus our knowledge about arclength to obtain a more practical formulation

$$\mathbb{K}(t) \equiv \frac{d\phi/dt}{ds/dt} \underset{\substack{\text{PROP.} \\ ②⑦③}}{=\!=} \frac{|dT/dt|}{ds/dt} = \left|\frac{dT}{ds}\right|$$

Since for arclength parametrization $ds/ds = 1$. Thus we may use,

$$\boxed{\mathbb{K}(t) = \frac{|T'(t)|}{|T(t)|}} \quad \boxed{|T'(s)| = \mathbb{K}(s)}$$

Remark: #46 of §10.3 in Stewart is intended to make more explicit the $d\phi/ds$ idea. You might notice Stewart only says in words what we have derived to prove the prop on ②⑦③. I'm following Colley's §3.2 pretty closely here. She has a bit more detail then Stewart offers. (as usual)

E37 The circle $r(t) = \langle a\cos t, a\sin t \rangle$, $0 \le t \le 2\pi$, assume $a > 0$,

$r'(t) = \langle -a\sin t, a\cos t \rangle$ $\therefore$ $|r'(t)| = \frac{ds}{dt} = \sqrt{a^2} = a$
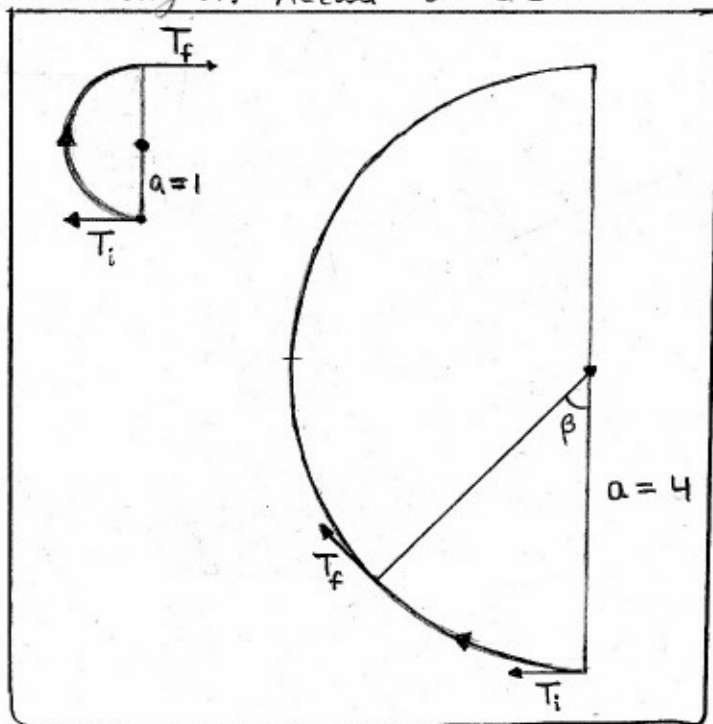
Thus the unit tangent,

$$T(t) = \frac{r'(t)}{|r'(t)|} = \langle -\sin t, \cos t \rangle$$

$$T'(t) = \langle -\cos t, -\sin t \rangle \quad \therefore \quad |T'(t)| = 1.$$

$$\therefore \quad \mathbb{K}(t) = \frac{|T'(t)|}{ds/dt} = \boxed{\frac{1}{a} = \mathbb{K}}$$: the curvature of a circle is inversely proportional to the radius of the circle.

I'll endeavor to sketch this, its clear that smaller circles force the T-vector to turn all the way around quicker for a given arclength. Recall $s = a\Theta$ need $4\beta = \pi \Rightarrow \beta = \pi/4$ on big circle gives same arc length.

you can clearly see that the Tangent vector completely reverses direction for the circle of $R = 1$ whereas for the larger circle of radius $R = 4$ the Tangent vector only changes direction by $\pi/4$ relative to it's initial state.

E38 Consider the line $r(t) = at + b$ for a, b fixed vectors. Then $r'(t) = a$ thus $T(t) = \frac{a}{|a|}$ hence $T'(t) = 0$. Consequently

$$\mathbb{K}(t) = \frac{|T'(t)|}{ds/dt} = \boxed{0 = \mathbb{K}}$$

lines have no curvature. One might also anticipate this result from E37 taking the radius $a \to \infty \Rightarrow \frac{1}{a} \to 0$.

The helix, $r(t) = \langle a\cos t, a\sin t, bt \rangle$. Recall we found in E36 that

$$T(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\sin t, a\cos t, b \rangle \quad \text{and} \quad |r'(t)| = \frac{ds}{dt} = \sqrt{a^2+b^2}$$

Calculate then,

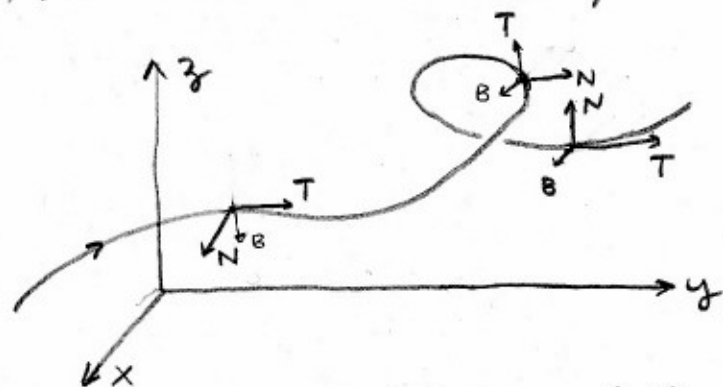$$T'(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\cos t, -a\sin t, 0 \rangle$$

$$|T'(t)| = \frac{1}{\sqrt{a^2+b^2}} \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\therefore \quad \kappa(t) = \frac{|T'(t)|}{ds/dt} = \frac{a/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \boxed{\frac{a}{a^2+b^2} = \kappa}$$

Again when $b = 0$ we find $\kappa = a/a^2 = 1/a$ which is consistent with what we found for the circle in E37.

$\{T, N, B\}$ & THE OSCULATING PLANE

Suppose that we have a path r such that $r'(t) \neq 0$ and $r'(t) \times r''(t) \neq 0 \; \forall t$. (its nonstop & not a line) then we can define three vectors which provide an orthogonal basis that moves with the path.

$$|T| = |N| = |B| = 1$$
(unit vectors)

The <u>osculating plane</u> has normal B and T, N actually reside in this plane. Moreover "locally" the osculating plane contains the path. Define,

$$\boxed{\; T(t) \equiv \frac{r'(t)}{|r'(t)|} \;\Bigg|\; N(t) \equiv \frac{T'(t)}{|T'(t)|} \;\Bigg|\; B(t) \equiv T(t) \times N(t) \;}$$

We proved $T \cdot N = 0$ essentially back in prop (i) on 273. The fact that $N \cdot B = 0$ and $T \cdot B = 0$ follows immediately from its def^n.

Proposition: $\dfrac{dT}{ds} = \mathbb{K}\,N$ for a nonlinear nonstop path.

Proof: not much to say here. Just recall definitions, and (274),

$$N = \frac{T'(s)}{|T'(s)|} \quad \text{and} \quad \mathbb{K} = \left|\frac{dT}{ds}\right| = |T'(s)|$$

$$\therefore N = \frac{T'(s)}{\mathbb{K}} \quad \Rightarrow \quad T'(s) = \boxed{\frac{dT}{ds} = \mathbb{K}\,N} \;//.$$

Notice we must avoid lines since $\mathbb{K} = 0$ for lines!

Proposition: $\dfrac{dB}{ds} \cdot T = 0$ so $\dfrac{dB}{ds} = -\tau\,N$ for some scalar function τ

Proof: see hwk sol^n §10.3 #47. I rehash much of the material here and give an explicit careful proof of this Prop. //

We now may be certain the def^n below is logical,

Def^n/ The torsion $\tau(s)$ is the function of arclength such that

$$\frac{dB}{ds} = -\tau\,N$$

geometrically the torsion measures how the curve twists out of the osculating plane. Thus for a curve which lies in a plane one finds the torsion is zero. Before any examples lets complete the theory, the following formulas were discovered about 1850 by Frédéric-Jean Frenet and Joseph Alfred Serret. They're known as the Frenet-Serret formulas. (See hwks # 47 & 48 of §10.3 for proof.)

$\dfrac{dT}{ds} = \mathbb{K}\,N$	$\dfrac{dN}{ds} = -\mathbb{K}\,T + \tau\,B$	$\dfrac{dB}{ds} = -\tau\,N$

Finally I quote a Th^m from Colley (2.5) which goes to show just how fundamental the quantities $\mathbb{K}$ and τ are to describing a curve.

Th^m/ Let C_1 and C_2 be smooth curves in $\mathbb{R}^3$ both with strictly positive curvatures $\mathbb{K}_1$ and $\mathbb{K}_2$. Then if $\mathbb{K}_1(s) = \mathbb{K}_2(s)$ and $\tau_1(s) = \tau_2(s)$ $\forall s$ then the curves C_1 & C_2 are congruent in the sense of highschool geometry ($C_1 = C_2 + b \leftarrow$ some fixed vector). Moreover the converse is true, given a positive arclength function and torsion one can uniquely reconstruct a curve upto translations.

E40 Consider the circle, $r(t) = \langle a\cos t, a\sin t, 0 \rangle$. We found in **E37** that
$T(t) = \langle -\sin t, \cos t, 0 \rangle$ and $T'(t) = \langle -\cos t, -\sin t, 0 \rangle$ so $|T'(t)| = 1$
thus we find $N(t) = T'(t) = \langle -\cos t, -\sin t, 0 \rangle$. Notice that

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, \sin^2 t + \cos^2 t \rangle = \langle 0, 0, 1 \rangle.$$

Thus B is a constant vector which means we get $\boxed{B(s) = \langle 0, 0, 1 \rangle}$,

$$\frac{dB}{ds} = 0 = -\tau N \quad \Rightarrow \quad \boxed{\tau = 0}$$

This is good, we predicted that planar curves (like a circle) have zero torsion. We should note that usually we will need to reparametrize $B(t)$ to $B(s)$ to find τ, this case was special.

E41 The helix $r(t) = \langle a\cos t, a\sin t, bt \rangle$ $\left(\text{see } \boxed{E33} \boxed{E34} \boxed{E36}, \boxed{E39} \right.$
$\left. \text{to find } T(t) \text{ and } T'(t) \right)$
$$T(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\sin t, a\cos t, b \rangle$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \left(\frac{\sqrt{a^2+b^2}}{a} \right) \frac{1}{\sqrt{a^2+b^2}} \langle -a\cos t, -a\sin t, 0 \rangle = \overbrace{\langle -\cos t, -\sin t, 0 \rangle}^{(**)}.$$

$$T(t) \times N(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\alpha a\sin t & \alpha a\cos t & \alpha b \\ -\cos t & -\sin t & 0 \end{vmatrix} \qquad \left(\begin{array}{l} \text{I let } \alpha \equiv \frac{1}{\sqrt{a^2+b^2}} \\ \text{because I'm tired of} \end{array} \right)$$

$$= \langle \alpha b \sin t, -\alpha b \cos t, \alpha a \sin^2 t + \alpha a \cos^2 t \rangle$$

$$= \alpha \cdot \langle b\sin t, -b\cos t, a \rangle = B(t)$$

We found in **E34** on (271) that $t = s/\sqrt{a^2+b^2} = s\alpha$ thus,

$$B(s) = \alpha \langle b\sin(\alpha s), -b\cos(\alpha s), a \rangle$$

$$\therefore \frac{dB}{ds} = \alpha^2 \langle b\cos(\alpha s), b\sin(\alpha s), 0 \rangle$$

$$= -b\alpha^2 \langle -\cos(\alpha s), -\sin(\alpha s), 0 \rangle : \text{ this is (**) reparam.}$$
with $t = \alpha s$. This
$$= -\tau N \qquad \qquad \text{is } N(s).$$

Comparing we find $\tau = b\alpha^2 = \boxed{\frac{b}{a^2+b^2} = \tau}$

Notice that as $b \rightarrow 0$ the torsion goes to zero, which is in agreement with **E40**.

E42 Consider $r(t) = \langle e^t\cos t, e^t\sin t, e^t \rangle$. Calculate the T, N, B frame and the curvature and torsion. We begin,

$$r'(t) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$$

$$r'(t) \cdot r'(t) = (e^t)^2(\cos t - \sin t)^2 + (e^t)^2(\sin t + \cos t)^2 + (e^t)^2$$

$$= e^{2t}[\cos^2 t - 2\sin t\cos t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t + 1]$$

$$= 3e^{2t} \quad \Rightarrow \quad \underline{|r'(t)| = \sqrt{3}\, e^t}$$

Thus the unit tangent vector $T(t) = r'(t)/|r'(t)|$ is

$$\boxed{T(t) = \tfrac{1}{\sqrt{3}}\langle \cos t - \sin t, \sin t + \cos t, 1 \rangle}$$

Recall that the chain rule says $\dfrac{dT}{dt} = \dfrac{ds}{dt}\dfrac{dT}{ds}$ ∴ $\dfrac{dT}{ds} = \dfrac{dT/dt}{ds/dt}$
But we know $\dfrac{ds}{dt} = |r'(t)| = \sqrt{3}e^t$ thus we find $\dfrac{dT}{ds}$ as a function of t,

$$\frac{dT}{ds} = \frac{1}{3e^t}\langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle$$

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{3e^t}\sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2} = \boxed{\frac{\sqrt{2}}{3e^t} = \kappa(t)}$$

I calculate the Normal vector N using an indirect method,

$$N = \frac{1}{\kappa}\frac{dT}{ds} = \frac{3e^t}{\sqrt{2}}\frac{1}{3e^t}\langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle$$

$$\underbrace{\phantom{N = \frac{1}{\kappa}\frac{dT}{ds}}}_{\text{see prop. on } 277} \qquad \therefore \boxed{N = \frac{1}{\sqrt{2}}\langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle}$$

And now the binormal follows from straightforward computation of $B = T \times N$ which yields, (I leave it for you)

$$\boxed{B = \frac{1}{\sqrt{6}}\langle \sin t - \cos t, -\sin t - \cos t, 2 \rangle}$$

Now we may deduce the torsion, again use the chain rule trick,

$$\frac{dB}{ds} = \frac{1}{ds/dt}\frac{dB}{dt} = \frac{1}{\sqrt{3}e^t}\frac{1}{\sqrt{6}}\langle \cos t + \sin t, -\cos t + \sin t, 0 \rangle$$

$$= \underbrace{\left(\frac{-1}{3e^t}\right)}_{-\tau}\underbrace{\frac{1}{\sqrt{2}}\langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle}_{N}$$

$$\boxed{\tau = \frac{1}{3e^t}}$$

Remark: the chain rule has saved us the trouble of computing the arclength!