To do it correctly we should give ε-δ formulation to make the limiting process precise. Take ma 425-426 if you're interested. We will continue in the heuristic tradition of ma 141, 241, all the elementary functions as well as their sums/differences/quotients/composites/roots are continuous where they are defined. No division by zero or $\sqrt{\text{negative}}$ # allowed. For $a \in \text{dom}(f)$

$$\boxed{\lim_{x\to a} f(x) = f(a) \Leftrightarrow f \text{ continuous at } a}$$

CONTINUITY & LIMITS

If $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ then f is continuous at (a,b) iff

$$\boxed{\lim_{(x,y)\to(a,b)} (f(x,y)) = f(a,b). \Leftrightarrow f \underset{\text{at } (a,b).}{\text{continuous}} \Leftrightarrow f \underset{C^0 \text{ at } (a,b)}{\text{is class}}}$$

Now this begs an obvious question, what do I mean by $(x,y) \longrightarrow (a,b)$? In the case $x \longrightarrow a$ recall we req^{d} $x \longrightarrow a^+$ <u>and</u> $x \longrightarrow a^-$. For two or more variables we need that $(x,y) \longrightarrow (a,b)$ from all directions and paths. If the value of $f(x,y)$ is approaching the same value for every direction then we call that limiting value $\lim_{(x,y)\to(a,b)} f(x,y)$. Just as $a \notin \text{dom}(f)$ necessarily we may also have $(a,b) \notin \text{dom}(f)$. In fact its only really interesting in those cases, otherwise why bother with the limit?

$\boxed{\text{E46}}$ Let $f(x,y) = x^2 + \sqrt{y} + \tan^{-1}(x) + 3$, find limit at $(0,1)$

$$\lim_{(x,y)\to(0,1)} \left(x^2 + \sqrt{y} + \tan^{-1}(x) + 3\right) = \underbrace{0 + \sqrt{1} + \tan^{-1}(0) + 3}_{\text{all the functions involved are well behaved at } x=0,\ y=1.} = \boxed{4}$$

TWO PATH TEST FOR NONEXISTENCE OF A LIMIT:

If a function $f(x,y)$ has limit L_1 along one path $(x,y) \xrightarrow{P_1} (a,b)$ and $L_2 \neq L_1$ along another path P_2 also approaching (a,b) then

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad \text{does not exist}$$

However if $L_1 = L_2$ then we can't conclude anything imediately, remember we have to approach the same value for all paths.

E47 Consider $f(x,y) = \begin{cases} 2xy/(x^2+y^2) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

this function is continuous everywhere except the origin, lets see why. Approach $(0,0)$ along the line $Y = mX$. As $(x,y) = (x,mx) \to (0,0)$,

$$\frac{2xy}{x^2+y^2} \longrightarrow \frac{2mx^2}{x^2+m^2x^2} \longrightarrow \frac{2m}{1+m^2}$$

Clearly for different lines we obtain different limits thus the limit d.n.e.

- Usually if you try $(x,y) = (x,0)$ or $(x,y) = (0,y)$ or $(x,y) = (x,x)$ then compare it will expose the limit's non-existance. There are more subtle cases,

E48 Consider $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$. Show $f \to 0$ as $(x,y) \to 0$.

First we try approaching via the line $Y = mx$, $(m \neq 0)$

$$\frac{2x^2y}{x^4+y^2} \longrightarrow \frac{2mx^3}{x^4+m^2x^2} \longrightarrow \frac{2mx}{x^2+m^2} \to \frac{2mx}{m^2} \to 0$$

On all nonhorizontal lines we find limit zero. This function is particularly sneaky. We consider a parabolic path $y = kx^2$

$$\frac{2x^2y}{x^4+y^2} \longrightarrow \frac{2kx^4}{x^4+k^2x^4} = \frac{2kx^4}{x^4(1+k^2)} \longrightarrow \frac{2k}{1+k^2} \neq 0 \quad \text{for } k \neq 0.$$

Thus the limit does not exist by the two path test.

Remark: I find it remarkable that we can prove many two-dimensional limits do exist. Its hard to imagine infinitely different paths. Obviously there is something subtle here, to really do it right take ma 426. We will content ourselves with this brief intro. and now move on to more practicle topics. You can see Stewart for more exampleo. I've borrowed these from the excellent calculus text by Thomas (10^{th} ed.) (§ 11.2)

Partial Differentiation

For a function of two independent variables x and y we define,

Def^n/ The partial derivative of $f(x,y)$ with respect to x at the point (x_o, y_o) is (provided the limit below exists)

$$\frac{\partial f}{\partial x}\bigg|_{(x_o,y_o)} \equiv \frac{d}{dx}\Big[f(x,y_o)\Big]\bigg|_{x=x_o} = \lim_{h\to 0}\left[\frac{f(x_o+h,y_o)-f(x_o,y_o)}{h}\right]$$

also we consider the partial derivative w.r.t. x as a function in its own right with values given in the obvious way.

$$\frac{\partial f}{\partial x}(x,y) \equiv \frac{\partial f}{\partial x}\bigg|_{(x,y)} \equiv f_x = z_x = \frac{\partial z}{\partial x}$$

where the last two notations are appropriate when considering $z = f(x,y)$.

Care to guess what $\partial f/\partial y$ means? Its the same,

Def^n/ We define the partial derivative w.r.t y at (x_o, y_o) to be

$$\frac{\partial f}{\partial y}\bigg|_{(x_o,y_o)} \equiv \frac{d}{dy}\Big[f(x_o,y)\Big]\bigg|_{y=y_o} = \lim_{h\to 0}\left[\frac{f(x_o,y+h)-f(x_o,y_o)}{h}\right]$$

we also consider the partial derivative w.r.t. y as a function in its own right, the values given pointwise by what we just def^d,

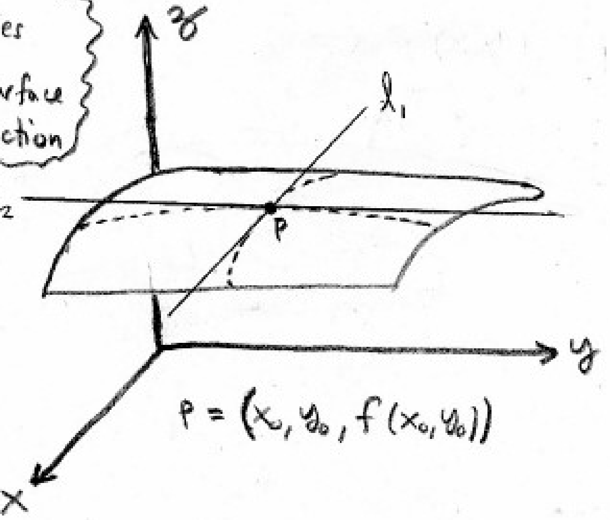
$$\frac{\partial f}{\partial y}(x,y) \equiv \frac{\partial f}{\partial y}\bigg|_{(x,y)} \equiv f_y = z_y = \frac{\partial z}{\partial y}$$

where the last two are appropriate for $z = f(x,y)$.

Partial Derivatives have a nice geometric meaning. The dotted lines indicate

$$z = f(x, y_o)$$
$$z = f(x_o, y)$$

$\frac{\partial f}{\partial x}\Big|_{(x_o,y_o)}$ gives slope of surface in x-direction



you could consider them functions of one variable on $y = y_o$ and $x = x_o$. In other words the dotted lines are the intersection curves of $z = f(x,y)$ with $x = x_o$ & $y = y_o$. The lines l_1 & l_2 lift off the surface are tangent to those curves.

If the eq^n $z = f(x,y)$ was a plane then l_1 and l_2 would actually reside on the plane. Geometrically we say the plane that passes through $(x_o, y_o, f(x_o,y_o))$ and just kisses $z = f(x,y)$ is the <u>tangent</u> plane. We'll find a more technical description on 306

Essentially the tangent plane is the best linear approx. to $z = f(x,y)$

Remark: A function $f(x,y)$ is differentiable at (a,b) if it has a unambiguous tangent plane at $(a,b,f(a,b))$. There are functions for which f_x and f_y exist at (a,b) yet there is no tangent plane. I turns out we need f_x & f_y to be continuous at (a,b) to insure differentiability of f. We denote all functions diff. at (a,b) by $C^1(a,b)$, for more skip to 306. For now we consider the basics,

E49 $F(x,y) = x^2 + y^2$.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left[x^2+y^2\right] = \frac{\partial}{\partial x}\left[x^2\right] + \frac{\partial}{\partial x}\left[\cancel{y^2}\right]^{\to 0} = \boxed{2x}$$

: y is constant with respect to x

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[x^2+y^2\right] = \boxed{2y}$$

: we regard x as constant as we perform the $\partial/\partial y$ operation.

E50 $F(x,y) = xe^{xy}$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy}\right) = \frac{\partial x}{\partial x}e^{xy} + x\frac{\partial}{\partial x}\left(e^{xy}\right)$$

: product rule

$$\boxed{F_x = e^{xy} + xye^{xy}}$$

: chain rule, remember y is regarded constant

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(xe^{xy}\right) = \underbrace{x\frac{\partial}{\partial y}\left(e^{xy}\right) = xe^{xy}\frac{\partial}{\partial y}(xy)}_{} = \boxed{x^2e^{xy} = F_y}$$

here I wrote out the chain rule, not always need, but may help in messy cases.

E51 $z^2 = \sin(xy) + x + \ln(y)$. Suppose that x & y are independent and z is dependent ; $z = z(x,y)$. We use implicit differention to find implicit formulas for z_x and z_y.

$$\frac{\partial}{\partial x}\left[z^2\right] = 2z\frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}\left[\sin(xy) + x + \ln(y)\right] = y\cos(xy) + 1$$

But these are equal, so $\boxed{\frac{\partial z}{\partial x} = \frac{1}{2z}\left[y\cos(xy)+1\right]}$

question: why is this implicit?

Likewise we calculate,

$$2z\frac{\partial z}{\partial y} = x\cos(xy) + \frac{1}{y} \quad \therefore \quad \boxed{\frac{\partial z}{\partial y} = \frac{1}{2z}\left[x\cos(xy) + \frac{1}{y}\right]}$$

PARTIAL DERIVATIVES OF $f(x_1, x_2, \dots, x_n)$:

The def$^{\underline{n}}$ of $\frac{\partial f}{\partial x_k}$ is essentially the same as that for $f(x,y)$, the meaning is that we take the ordinary derivative w.r.t. x_k while holding all the other variables fixed. That is,

$$\frac{\partial f}{\partial x_k}(x_1, x_2, \dots, x_n) \equiv \lim_{h \to 0}\left(\frac{f(x_1, x_2, \dots, x_k + h, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h}\right)$$

We also may employ the notations,

$$\frac{\partial f}{\partial x_k} \equiv \partial_k f \equiv f_{x_k}$$

$\boxed{E52}$ Let $g(x,y,z) = xy^2z^3 + \sin(xyz)$ then

$$g_x = y^2z^3 + yz\cos(xyz) \quad : \ y, z \text{ treated as constants.}$$

$$g_y = 2xyz^3 + xz\cos(xyz) \quad : \ x, z \text{ treated as constants.}$$

$$g_z = 3xy^2z^2 + xy\cos(xyz) \quad : \ x, y \text{ treated as constants.}$$

$\boxed{E53}$ Suppose $r = \sqrt{x^2 + y^2 + z^2}$.

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\left[x^2+y^2+z^2\right] = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Likewise $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$.

Remark: the hwk sol$^{\underline{n}}$ has many more examples.

HIGHER PARTIAL DERIVATIVES:

have the obvious meaning, we simply iterate. For examples,

$\boxed{E54}$ Let $f(x,y) = xy^2$.

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[y^2\right] = 0.$$

$$f_{yx} \equiv \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[2xy\right] = 2y.$$

$$f_{xy} \equiv \frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[y^2\right] = 2y.$$

interesting, is it always the case that $f_{xy} = f_{yx}$?

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[2yx\right] = 2x.$$

- you may consult your hwk for added examples on this topic.

A curious Example: Why $f_{xy} \neq f_{yx}$ always.

$$f(x,y) = \begin{cases} (x^3y - xy^3)/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

When $(x,y) \neq (0,0)$ its a simple matter to differentiate,

$$f_x = \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y = \frac{x^5 - 4y^2x^3 - y^4x}{(x^2+y^2)^2}$$

$$f_{xy} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = f_{yx}(x,y) \quad \text{for } (x,y) \neq 0.$$

At the origin we need to use the def$^{\underline{n}}$ of partial differentiation,

$$f_x(0,0) = \lim_{h\to 0}\left[\frac{f(h,0) - f(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{0-0}{h}\right] = 0.$$

$$f_y(0,0) = \lim_{h\to 0}\left[\frac{f(0,h) - f(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{0-0}{h}\right] = 0.$$

$$f_{xy}(0,0) \equiv \frac{\partial f_x}{\partial y}(0,0) = \lim_{h\to 0}\left[\frac{f_x(0,h) - f_x(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{-h^5/(h^2)^2 - 0}{h}\right] = -1.$$

$$f_{yx}(0,0) \equiv \frac{\partial f_y}{\partial x}(0,0) = \lim_{h\to 0}\left[\frac{f_y(h,0) - f_y(0,0)}{h}\right] = \lim_{h\to 0}\left[\frac{h^5/h^4 - 0}{h}\right] = 1.$$

Therefore $f_{xy} \neq f_{yx}$ since at $(0,0)$ they disagree. You might object that this to picky on our part, well sorry its math. The trouble here is that f_{xy} is not continuous at $(0,0)$, everywhere else it is and in all those places $f_{xy}(x,y) = f_{yx}(x,y)$ $\forall (x,y) \neq (0,0)$.

> **Clairaut's Th$^{\underline{m}}$:** Suppose f is defined on some disk containing (a,b). If the functions f_{xy} and f_{yx} are both continuous on D then
> $$f_{xy}(a,b) = f_{yx}(a,b)$$

Proof: you can look it up in the appendix, or find a more advanced text perhaps. Anyway you can see from the counterexample given above that it takes a fairly contrived function to escape the usual fact that $f_{xy} = f_{yx}$.

Remark: we have covered §11.2 and §11.3 approximately on pgs. 290 → 295. Next we discuss the chain rule for several variables, after that I include some material on constrained partials (seemingly not in Stewart) then we will study the tangent plane and linearization (§11.4 + §11.6).

THE CHAIN RULE FOR FUNCTIONS OF SEVERAL VARIABLES

To begin we will treat the simple cases where our intermediate variables are independent, later we'll deal with some subtler cases which arise in common applications. First the basics. Assume that f is differentiable in each of the following.

① If $w = f(x)$ and $x = x(t)$ then:

$$\frac{dw}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

② If $w = f(x,y)$ and x, y are functions of t then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

③ If $w = f(x,y,z)$ and x, y, z are functions of t then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

④ If $w = f(x_1, x_2, \dots, x_n)$ and $x_1, x_2, \dots, x_n$ are functions of t then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

We know ① to be true from calculus I. Lets give a "proof" of ② since the proofs of ③ & ④ are the same with a bit more writing. We have yet to define differentiability but we can essentially take it to mean that f is approximated by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

so for (x,y) "close" to (a,b) we have $f(x,y) \cong L(x,y)$. We'll discuss the linearization $L(x,y)$ more later on. For now, consider if $w = f(x,y)$ and let $(x(t_0), y(t_0)) = (a,b)$.

$$\begin{aligned}
\left.\frac{dw}{dt}\right|_{t_0} &= \frac{d}{dt}\Big[f(x(t),y(t))\Big]\Big|_{t_0} \\
&= \frac{d}{dt}\Big[f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)\Big]\Big|_{t_0} \\
&= f_x(a,b)\left.\frac{dx}{dt}\right|_{t_0} + f_y(a,b)\left.\frac{dy}{dt}\right|_{t_0}
\end{aligned}$$

$$\therefore \frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t),y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t))\frac{dy}{dt}.$$

A good technical proof needs ε's and δ's. Ask me if your interested or take ma 426.

Examples of $f(x,y)$ or $f(x,y,z)$ where the intermediate variables x, y, z are themselves functions of an independent variable t.

E55 Let $w = xy$ and suppose $x = e^t$ and $y = \sin t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = ye^t + x\cos t = \boxed{e^t \sin t + e^t \cos t = \frac{dw}{dt}}$$

E56 Let $z = xy$ and again suppose $x = e^t$ & $y = \sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \boxed{e^t \sin t + e^t \cos t = \frac{dz}{dt}}$$

- Sometimes z is taken as the dependent variable. Other times z is playing the role of an intermediate variable.

E57 Let $w = xyz$ and suppose $x = t$, $y = t^2$, $z = t^3$

$$\frac{dw}{dt} = \frac{d}{dt}\left[w(x(t), y(t), z(t)) \right]$$

$$= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= yz + xz(2t) + xy(3t^2)$$

$$= t^5 + 2t^5 + 3t^5$$

$$= \boxed{6t^5 = \frac{dw}{dt}}$$ (which is good since $w = t^6$ so $\frac{dw}{dt} = 6t^5$)

Some people find the following "Tree Diagrams" a help in figuring out the correct chain rule. You may use them if they help

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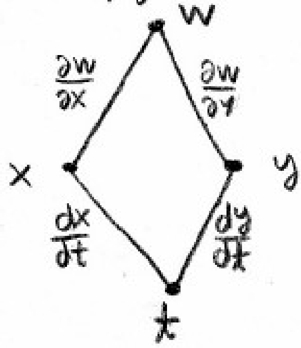
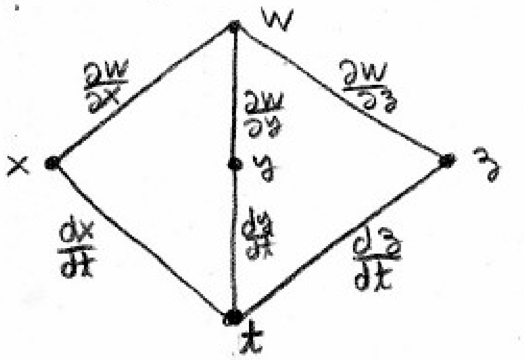
You multiply down then add together,





dependent

intermediate

independent

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Again assume that f is differentiable, also suppose the intermediate variables x, y, z are functions of s, t (or u, v or $u_1, u_2, \dots, u_k$)

⑤. If $w = f(x)$ and $x = x(s,t)$ then:

$$\frac{\partial w}{\partial s} = \frac{df}{dx}\frac{\partial x}{\partial s} \quad \& \quad \frac{\partial w}{\partial t} = \frac{df}{dx}\frac{\partial x}{\partial t}$$

⑥ If $w = f(x,y)$ and x, y are functions of s & t then:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \& \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

⑦ If $w = f(x,y,z)$ and x, y, z are functions of s, t then:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \quad \& \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

⑧ If $w = f(x_1, x_2, \dots, x_n)$ and $x_1, x_2, \dots, x_n$ are functions of s, t then:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial s} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t}$$

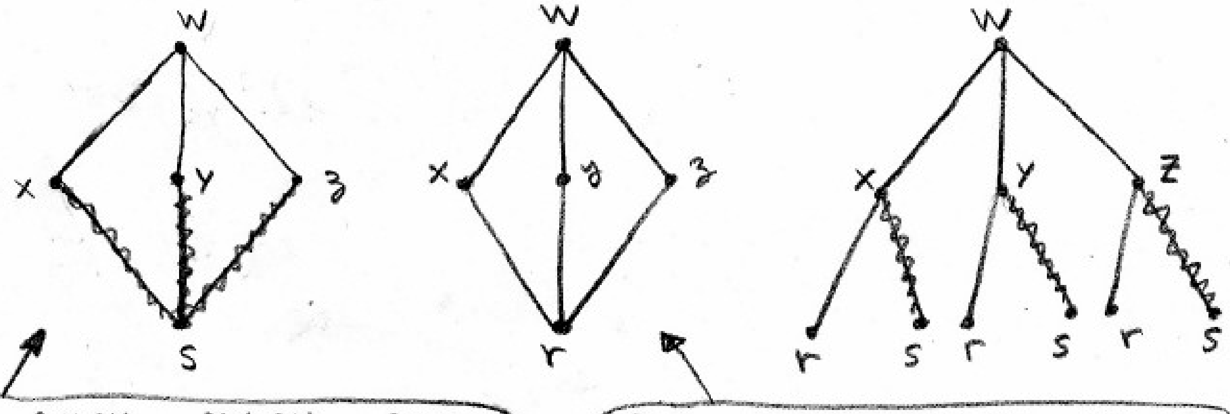
⑨ If $w = f(x_1, x_2, \dots, x_n)$ and $x_1, \dots, x_n$ are fncts. of $u_1, u_2, \dots, u_k$ then:

$$\frac{\partial w}{\partial u_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial u_1} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial u_1}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial w}{\partial u_k} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial u_n} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial u_k} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial u_k}$$

With Tree Diagrams: (I can't do color here so I'll use squiggles)



$$\boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}} \qquad \boxed{\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}}$$

Remark: The proof of 5 → 9 follows from the earlier results 1 → 4. You simply fix either s or t etc... and so the $\frac{d}{dt}$'s become $\frac{\partial}{\partial t}$'s. Remember the "$\partial$" notation just reminds us that there are possibly several variables which ride along unchanged. You can write for $f = f(x)$ that $\frac{df}{dx} = \frac{\partial f}{\partial x}$ if you wish. However if $f = f(x,y)$ then we must be careful to distinguish the concepts.

E58 Suppose $w = e^x \sin(y)$ and $y = st^2$, $x = \ln(s-t)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad : \text{identified that } w \text{ is function of } x \,\&\, y.$$

$$= e^x \sin(y)\frac{1}{s-t} + e^x \cos(y)\, t^2 \quad : \text{note } e^x = e^{\ln(s-t)} = s-t.$$

$$= \boxed{\sin(st^2) + (s-t)t^2\cos(st^2) = \frac{\partial w}{\partial s}}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$= e^x \sin(y)\left(\frac{-1}{s-t}\right) + e^x\cos(y)(2st) \quad ; \text{ again } e^x = s-t,$$

$$= \boxed{-\sin(st^2) + 2st(s-t)\cos(st^2) = \frac{\partial w}{\partial t}}$$

E59 Suppose that $z = f(x,y)$ has continuous $f_x, f_y, f_{xy}, f_{yx}, f_{xx}$ etc... and $x = r^2+s^2$ and $y = 2rs$ then note,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}$$

We wish to compute $\partial^2 z/\partial r^2$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right]$$

$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\right] + 2s\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\right]$$

$$= 2\frac{\partial z}{\partial x} + 2r\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}\right] + 2s\left[\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right]$$

$$= 2\frac{\partial z}{\partial x} + 2r\left[\frac{\partial^2 z}{\partial x^2}\cdot 2r + \frac{\partial^2 z}{\partial y\partial x}\cdot 2s\right] + 2s\left[\frac{\partial^2 z}{\partial x\partial y}\cdot 2r + \frac{\partial^2 z}{\partial y^2}\cdot 2s\right]$$

$$= 2f_x + 4r^2 f_{xx} + 4rs f_{xy} + 4sr f_{yx} + 4s^2 f_{yy}$$

$$= \boxed{2f_x + 4r^2 f_{xx} + 4s^2 f_{yy} + 8rs f_{xy}} \quad \text{using Clairaut's Th}^{\underline{m}}.$$

(This is the best we can do w/o an explicit formula for $f(x,y)$.)

Implicit Differentiation

Upto now we have considered x, y to be independent (or x, y, z) and then we supposed that x, y had parametrization in terms of one or two (or more) independent variables s, t. What if we take a different view, what if we assume that x, y, z are related implicitly through some relation. To begin we consider the problem : $F(x,y) = 0$ find $\frac{dy}{dx}$. Recall in calc. I we use implicit differentiation to do this for example,

$\boxed{E60}$ $\sin(x)\cos(Y) + Y^2 = x^3$: suppose $Y = Y(x)$ find $\frac{dY}{dx}$

$$\cos(x)\cos(Y) - \sin(x)\sin(Y)\frac{dY}{dx} + 2Y\frac{dY}{dx} = 3x^2$$

$$\therefore \frac{dY}{dx} = \frac{3x^2 - \cos(x)\cos(y)}{2Y - \sin(x)\sin(y)}.$$

We may arive at this result through another approach, perhaps easier.

① Set-up : Suppose $F(x, y) = 0$ implicitly defines $y = f(x)$ such that $F(x, f(x)) = 0$. Now differentiate w.r.t. x, (here "t" $= x$)

$$\frac{dF}{dx} = \frac{d}{dx}(0) = 0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial Y}\frac{dY}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial Y}\frac{dY}{dx} = 0$$

$$\therefore \boxed{\frac{dY}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial Y}}}$$ "eq$^{\underline{n}}$ 6" of Stewart, (pg. 785)

$\boxed{E61}$ Lets revisit $\boxed{E60}$ to begin define $F(x,Y) = x^3 - \sin(x)\cos(Y) - Y^2 = 0$,

$$\frac{dY}{dx} = \frac{-F_x}{F_Y} = \frac{-(3x^2 - \cos(x)\cos(Y))}{\sin(x)\sin(Y) - 2Y} = \frac{3x^2 - \cos(x)\cos(Y)}{2Y - \sin(x)\sin(Y)}.$$

(this can shortcut alot of calculation, not so much for my example).

② SET-UP : Suppose $F(x,y,z) = 0$ implicitly defines $z = f(x,y)$, so that $F(x,y,f(x,y)) = 0$. Then

$$\frac{dF}{dx} = 0 = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \quad \therefore \boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}}$$

Likewise we can derive that

$$\frac{dF}{dy} = 0 = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial y}}_{0} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial y}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} \quad \therefore \boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$

E62 Consider $x^2+y^2+z^2 = 1$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. We suppose that $z = z(x,y)$, define $F(x,y,z) = x^2+y^2+z^2-1$. Then,

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-2x}{2z} = \boxed{\frac{-x}{z} = \frac{\partial z}{\partial x}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-2y}{2z} = \boxed{\frac{-y}{z} = \frac{\partial z}{\partial y}}$$

Remark: The implicit function Th^{m} states that given certain conditions the eq^{n} $F(x,y) = 0$ or $F(x,y,z) = 0$ gives $y = y(x)$ or $z = z(x,y)$. That is one of the variables is given implicitly as a function of the remaining variables. The one "dependent" variable we took to be y in E61 or z in E62, but why not the other way around? Why isn't x the dependent variable? The answer, it can be. This concept is not just a mathematical digression, there are common engineering / physics applications for which a given set of variables are w/o a preference for choosing "dependent" variable. Probably the most familiar is the eq^{n}

$$PV = nRT$$

here P, V, T are all variables, which one is the dependent? I'll spend a few pages exposing the danger and introducing a notation to fix the potential pitfall.

(We are indebted to Thomas' Calc. § 11.9)

Partial Derivatives of Constrained Variables

This does not seem to appear explicitly in Stewart. I borrow the presentation of Thomas' Calc. 10th Ed. §11.9. The topic here has the potential to be confusing but with proper notation the trouble can be avoided. This notation is absent from Stewart see problems 72, 73, 75 of §11.3 and #47 of §11.5, he takes an indirect approach by mentioning the independent/dependent variables in the set-up for the problem. I think its better to be a bit more explicit about this issue. Lets begin with what can go wrong if we are not careful. (Also see p. 157 # 21 → 24 of Colley I've added those as recommended hwks)

Cautionary Example:

Find $\partial w/\partial x$ given that $w = x^2+y^2+z^2$ and $z = x^2+y^2$

We have two eq^{n}'s and four unknowns. Clearly since we are asked to find $\partial w/\partial x$ this suggests w is a dependent variable whereas x is independent. This leaves y and z, one of these must be dep. the other independent. That is we either write

$$w = w(x,y) \quad \text{or} \quad w(x,z)$$

Suppose $w = w(x, y, z(y))$: x, y are independent

$$\begin{aligned} w &= x^2+y^2+z^2 && : \text{need to leave just } x \,\&\, y \\ &= x^2+y^2+(x^2+y^2)^2 && : \text{substitute } z = x^2+y^2 \\ &= x^2+y^2+x^4+2x^2y^2+y^4 \end{aligned}$$

$$\therefore \boxed{\frac{\partial w}{\partial x} = 2x+4x^3+4xy^2}$$

Suppose $w = w(x, y(z), z)$: x, z are independent

$$\begin{aligned} w &= x^2+y^2+z^2 && : \text{need to eliminate } y \\ &= x^2+z-x^2+z^2 && : \text{solve } z = x^2+y^2 \text{ for } y^2 = z - x^2. \\ &= z+z^2 \end{aligned}$$

$$\therefore \boxed{\frac{\partial w}{\partial x} = 0}$$

!?☹ etc...

this is why the independent / dependent variables must be made explicit.

Notational Cure:	
$\left(\frac{\partial w}{\partial x}\right)_y = 2x+4x^3+4xy^2$	$\left(\frac{\partial w}{\partial x}\right)_z = 0$

Bonus Point: expose the geometric meaning of $\left(\frac{\partial w}{\partial x}\right)_y \neq \left(\frac{\partial w}{\partial x}\right)_z$ found in the "Cautionary Example". Its not that complicated we just don't have time for it.

ADVICE: How to find $\frac{\partial w}{\partial x}$ when $W = F(x, y, z)$ has inputs constrained by another equation

1. Decide which variables are dependent or independent,
2. Eliminate the other dependent variables in W.
3. Differentiate as usual.

• Let me further illustrate with some real-world examples

$\boxed{E63}$ Suppose that $PV = nRT$ where n, R are constants and P = pressure, V = volume, T = temperature. This is the Ideal Gas Law. Problem 72 of §11.2 asks us to show that $\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$. Lets clarify that, we need to show $\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_V = -1$.

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{\partial}{\partial V}\left[\frac{nRT}{V}\right]\Bigg|_{T \text{ fixed}} = nRT\left(\frac{-1}{V^2}\right) \quad : \quad P = P(V,T)$$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{\partial}{\partial T}\left[\frac{nRT}{P}\right]\Bigg|_{P \text{ fixed}} = \frac{nR}{P} \quad : \quad V = V(T,P)$$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{\partial}{\partial P}\left[\frac{PV}{nR}\right]\Bigg|_{V \text{ fixed}} = \frac{V}{nR}$$

Now assemble these and remember $PV = nRT$,

$$\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial T}{\partial P}\right)_V = \left(-\frac{nRT}{V^2}\right)\left(\frac{\cancel{nR}}{P}\right)\left(\frac{V}{\cancel{nR}}\right) = \frac{-nRT}{PV} = \frac{-nRT}{nRT} = -1.$$

Remark: this hwk problem is really confusing if you don't settle the question of whats a function of what! Trust me.

E64 Lets look at #75 of §11.3

Given $K = \frac{1}{2}mv^2$ show that $\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$. To begin lets make this problem statement more precise, show

$$\boxed{\left(\frac{\partial K}{\partial m}\right)_v \left(\frac{\partial^2 K}{\partial v^2}\right)_m = K}$$

very well, lets begin

$$\left(\frac{\partial K}{\partial m}\right)_v = \frac{\partial}{\partial m}\left[\frac{1}{2}mv^2\right]\Big|_{v\text{- fixed}} = \frac{1}{2}v^2$$

$$\left(\frac{\partial K}{\partial v}\right)_m = \frac{\partial}{\partial v}\left[\frac{1}{2}mv^2\right]\Big|_{m\text{-fixed}} = mv \quad \text{(momentum !)}$$

$$\left(\frac{\partial^2 K}{\partial v^2}\right)_m = \frac{\partial}{\partial v}\left[mv\right]\Big|_{m\text{-fixed}} = m.$$

Therefore we find that $K = \frac{1}{2}mv^2 = \boxed{\left(\frac{\partial K}{\partial m}\right)_v \left(\frac{\partial^2 K}{\partial v^2}\right)_m = K}$.

Remark: the interesting examples are found in thermodynamics. I'll put off those until we discuss total differentials.

E65 Again suppose $PV = NRT$, but this time suppose that only R is constant so P, V, N, T are variables.

$$\left(\frac{\partial P}{\partial T}\right)_{V,N} = \frac{\partial}{\partial T}\left[\frac{NRT}{V}\right]\Big|_{V,N \text{ fixed}} = \frac{NR}{V}$$

$$\left(\frac{\partial T}{\partial V}\right)_{P,N} = \frac{\partial}{\partial V}\left[\frac{PV}{NR}\right]\Big|_{P,N \text{ fixed}} = \frac{P}{NR}$$

$$\left(\frac{\partial V}{\partial N}\right)_{P,T} = \frac{\partial}{\partial N}\left[\frac{NRT}{P}\right]\Big|_{P,T \text{ fixed}} = \frac{RT}{P}$$

$$\left(\frac{\partial N}{\partial P}\right)_{T,V} = \frac{\partial}{\partial P}\left[\frac{PV}{RT}\right]\Big|_{T,V \text{ fixed}} = \frac{V}{RT}$$

$$\left(\frac{\partial P}{\partial T}\right)_{V,N}\left(\frac{\partial T}{\partial V}\right)_{P,N}\left(\frac{\partial V}{\partial N}\right)_{P,T}\left(\frac{\partial N}{\partial P}\right)_{T,V} = \frac{\cancel{NR}}{\cancel{V}}\frac{\cancel{P}}{\cancel{NR}}\frac{\cancel{RT}}{\cancel{P}}\frac{\cancel{V}}{\cancel{RT}} = 1.$$

Curious. I have no idea what this means. Enlighten me if you know.

$\boxed{E66}$ Suppose $U = f(P,V,T)$ = internal energy of a gas that obeys the Ideal Gas Law $PV = nRT$ (n, R constants).

$$\left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial}{\partial P}\Big[f(P,V,T)\Big]\Big|_{V\text{-fixed}} = \frac{\partial f}{\partial P}\frac{\partial P}{\partial P} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial P} = \boxed{\frac{\partial f}{\partial P} + \frac{\partial f}{\partial T}\frac{V}{nR}}$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial}{\partial T}\Big[f(P,V,T)\Big]\Big|_{V\text{-fixed}} = \frac{\partial f}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial f}{\partial T}\frac{\partial T}{\partial T} = \boxed{\frac{\partial f}{\partial P}\left(\frac{nR}{V}\right) + \frac{\partial f}{\partial T}}$$

I've used $T = PV/nR$ and $P = nRT/V$ to calculate $\partial T/\partial P$ & $\partial P/\partial T$. And its better to write $\frac{\partial U}{\partial P} + \frac{\partial U}{\partial T}\frac{V}{nR} = \left(\frac{\partial U}{\partial P}\right)_V$ and $\frac{\partial U}{\partial P}\frac{nR}{V} + \frac{\partial U}{\partial T} = \left(\frac{\partial U}{\partial T}\right)_V$.

$\boxed{E67}$ Suppose that $x^2 + y^2 = r^2$ and $x = r\cos\theta$, $y = r\sin\theta$,

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \frac{\partial}{\partial r}\Big[r\cos\theta\Big]\Big|_{\theta\text{ fixed}} = \cos\theta$$

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{\partial}{\partial x}\Big[r\Big]\Big|_{y\text{-fixed}} = \frac{\partial}{\partial x}\Big[\sqrt{x^2+y^2}\Big]\Big|_{y\text{-fixed}} = \frac{x}{\sqrt{x^2+y^2}}$$

Just trying to eulicidate the notation.

$\boxed{E68}$ Show if $f(x,y,z) = 0$ then $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1$. We begin by exploiting $f(x,y,z) = 0$ to give a few differential relations,

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \quad \therefore \left(\frac{\partial z}{\partial x}\right)_y = \frac{-\partial f/\partial x}{\partial f/\partial z} \quad \left(\frac{\partial y}{\partial x} = 0\right)$$

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} \quad \therefore \left(\frac{\partial x}{\partial y}\right)_z = \frac{-\partial f/\partial y}{\partial f/\partial x} \quad \left(\frac{\partial z}{\partial y} = 0\right)$$

$$\frac{\partial f}{\partial z} = 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z} \quad \therefore \left(\frac{\partial y}{\partial z}\right)_x = \frac{-\partial f/\partial z}{\partial f/\partial y} \quad \left(\frac{\partial x}{\partial z} = 0\right)$$

Thus,

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{-f_y}{f_x}\right)\left(\frac{-f_z}{f_y}\right)\left(\frac{-f_x}{f_z}\right) = -1.$$

<u>Remark</u>: hmm... this example seems quite close to one of the req^{d} homeworks. in §11.5. (#47)...

I pause to outline a few formal ideas that can help bring more clarity to our calculations. Read Colley §2.3, 2.4 and 2.5 for the whole story. Colley uses matrix notation so she can say far more general things efficiently. This material is not all req'd, I include it for breadth, so that you can be "well-rounded" (sorry I threw up a little just writing that ☺)

Def^{n}/ Let $\mathcal{X}$ be open in $\mathbb{R}^2$ and $f : \mathcal{X} \subseteq \mathbb{R}^2 \to \mathbb{R}$ then f is differentiable at $(a,b) \in \mathcal{X}$ if the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ exist and if the function

$$h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is a good linear approx. to f near (a,b). That is if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - h(x,y)}{|(x,y)-(a,b)|} = 0$$

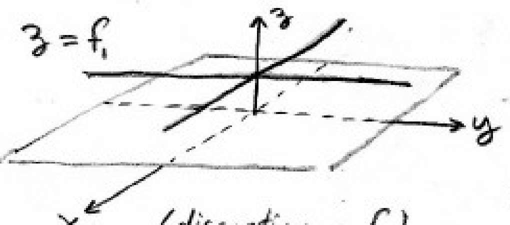
Moreover, if f is differentiable at (a,b) then the eq^{n} $z = h(x,y)$ <u>defines</u> the tangent plane to the graph of f at $(a,b,f(a,b))$. If f is diff. $\forall (a,b) \in \mathrm{dom}(f)$ then we say f is differentiable and write $f \in C^1(\mathcal{X}, \mathbb{R})$

You can compare this to Stewart's comments on p. 770 where he defines the tangent plane and Def^{n}(7) on p. 772 where he defines differentiability of f. This is simply a precise way of saying the same things.

Th^{m}/ Suppose $\mathcal{X}$ is open in $\mathbb{R}^2$. If $f : \mathcal{X} \to \mathbb{R}$ has <u>continuous</u> <u>partial</u> <u>derivatives</u> in a neighborhood of (a,b) in $\mathcal{X}$ then f is differentiable at (a,b).

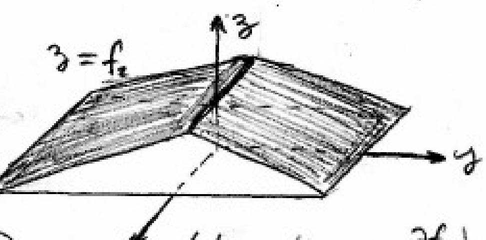
A function $f : \mathbb{R}^2 \to \mathbb{R}$ may fail to be differentiable due to discontinuity or discontinuous partials.

$$f_1(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases} \qquad\qquad f_2(x,y) = 1 - |y|$$



(discontinuous f)

No well-defined tangent plane because f_1 & $f_2 \notin C^1$



(discontinuous $\frac{\partial f}{\partial y}$)

Just as in the case $f: \mathbb{R} \to \mathbb{R}$ (calc. I) we can prove that,

$\boxed{\text{Th}^{\underline{m}}/ \text{ If } f: X \subseteq \mathbb{R}^2 \to \mathbb{R} \text{ is differentiable at } (a,b) \text{ then } f \text{ is continuous at } (a,b)}$

We may generalize the idea of differentiability to $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$. But first let me look ahead a bit and define the gradient

$$\nabla f = [\partial_1 f, \partial_2 f, \dots, \partial_n f] \quad \text{where} \quad \partial_i f \equiv \frac{\partial f}{\partial x_i}$$

$$(\nabla f)(\vec{a}) = [\partial_1 f(\vec{a}), \partial_2 f(\vec{a}), \dots, \partial_n f(\vec{a})] \quad \text{where} \quad \vec{a} = (a_1, a_2, \dots, a_n)^T$$

$$\underbrace{\nabla f(a) \bullet (\vec{x}-\vec{a})}_{} = [\partial_1 f(\vec{a}), \dots, \partial_n f(\vec{a})] \begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix} = \partial_1 f(\vec{a})(x_1 - a_1) + \dots + \partial_n f(\vec{a})(x_n - a_n).$$

dot-product in $\mathbb{R}^n$, can be encoded by matrix multiplication
$x \cdot y = x^T y$

Def$^{\underline{n}}$/ Let X be open in $\mathbb{R}^n$ and $f: X \to \mathbb{R}$ and take a point $\vec{a} = (a_1, a_2, \dots, a_n)^T \in X$. We say f is differentiable at $\vec{a}$ if all the partials $\partial_i f(\vec{a})$ exist and if $h: \mathbb{R}^n \to \mathbb{R}$ def$^{\underline{n}}$ by

$$h(\vec{x}) = f(\vec{a}) + [\nabla f(\vec{a})] \bullet [\vec{x} - \vec{a}]$$

is a good linear approximation to f near $\vec{a}$, meaning,

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - h(\vec{x})}{|\vec{x} - \vec{a}|} = 0$$

Again the condition of differentiability amounts to the existence of a <u>best linear approximation</u> ($h(\vec{x})$). This is in turn equivalent to the existence of a well-defined hyperplane to the graph $x_{n+1} = f(x_1, x_2, \dots, x_n)$ (a hypersurface in $\mathbb{R}^{n+1}$). The eq$^{\underline{n}}$ to the <u>tangent hyperplane</u> at $(\vec{a}, f(\vec{a}))$ is,

$$x_{n+1} = h(\vec{x}) = f(\vec{a}) + (\nabla f)(\vec{a}) \bullet (\vec{x} - \vec{a})$$

<u>Remark</u>: $z = f(a,b) + (\partial_x f)(a,b)(x-a) + (\partial_y f)(a,b)(y-b)$ and $y = f(a) + f'(a)(x-a)$ are the cases $n=2$ and $n=1$ where the "tangent hyperplanes" are actually an ordinary plane and a line.

The Jacobean Matrix

We bring our tour of the theory of multivariate differentiation to the most general case. We consider $f: \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, this is an m-dim'l vector-valued function of n-variables $x_1, x_2, \ldots, x_n$, we can express f in terms of its component functions f_j,

$$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))^T \in \mathbb{R}^m$$

Then the matrix $Df(x_1, x_2, \ldots, x_n)$ is the Jacobean Matrix of f,

$$Df(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} (\nabla f_1)^T \\ (\nabla f_2)^T \\ \vdots \\ (\nabla f_m)^T \end{bmatrix}$$

the "T" is for transpose it makes the column ∇f into a row $(\nabla f)^T$.

Def^n/ Let $\mathbb{X} \subseteq \mathbb{R}^n$ be open and let $f: \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Pick $\vec{a} \in \mathbb{X}$ then f is differentiable at $\vec{a}$ if $Df(\vec{a})$ exists and if the function $h: \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$h(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

is a good linear approx. to f near $\vec{a}$. That is

$$\lim_{\vec{x} \to \vec{a}} \frac{|f(x) - h(x)|}{|x - a|} = \lim_{x \to a} \frac{|f(x) - f(a) - Df(\vec{a})(\vec{x} - \vec{a})|}{|x - a|} = 0$$

So we finally arive at the general idea of differentiation. The derivative of a function is the <u>best</u> <u>linear</u> <u>approximation</u>. That is if $f: \mathbb{X} \subset \mathbb{R}^n \to \mathbb{R}^m$ then f is diff. at $a \in \mathbb{X}$ if $\exists\, L: \mathbb{R}^n \to \mathbb{R}^m$ which is a <u>linear</u> <u>mapping</u> $(L(\alpha x + y) = \alpha L(x) + L(y))$ such that

$$\lim_{x \to a} \frac{|f(x) - f(a) + L(x - a)|}{|x - a|} = 0$$

<u>Remark</u>: If you find these generalities interesting you should take ma 426 where you'll prove many of these results. I include them because of the examples I'm about to give. You are only expected to get the big ideas here,

Properties of the "Derivative"

These results are quite general. We consider $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ then if f is differentiable the we say $Df : \mathbb{R}^n \to \mathbb{R}^m$ is the derivative of f. Supposing f & g are diff. from $\mathbb{R}^n \to \mathbb{R}^m$ ($f, g \in C'(\mathbb{R}^n, \mathbb{R}^m)$) then for $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$D(f+g)(a) = (Df)(a) + (Dg)(a).$$

$$D(cf)(a) = c\,Df(a).$$

I don't find these results particularly surprising, however the next property, the Generalized Chain Rule, I found suprisingly simple. Let $h = F \circ G$ where $F : \mathbb{X} \subseteq \mathbb{R}^m \to \mathbb{R}^p$ and $G : \mathbb{Y} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ so that $h : \mathbb{R}^n \xrightarrow{G} \mathbb{R}^m \xrightarrow{F} \mathbb{R}^p$ and suppose that $\vec{t} \in \mathbb{R}^n$ such that $h = F \circ G$ is differentiable at $\vec{t}$ then

$$[Dh](\vec{t}) = [D(F \circ G)](\vec{t}) = (DF)(G(\vec{t}))\, DG(\vec{t})$$

where there are matrix multiplications implicit in the above, if we use operator notation then $D(F \circ G) = DF \circ DG$. This encompasses **all** the various unconstrained chainrules we've detailed upto now. (See Colley §2.5 for more details)

$\boxed{E69}$ Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Define the polar coordinate change map; $\mathbb{X}(r,\theta) \equiv (r\cos\theta, r\sin\theta)$ this means $\mathbb{X} : \mathbb{R}^2 \to \mathbb{R}^2$ it takes $(r,\theta) \longmapsto (x(r,\theta), y(r,\theta))$. Consider $g = f \circ \mathbb{X}$. Then $Dg = Df \circ D\mathbb{X}$ where $\mathbb{X} = (x, y)$

$$Df = \left[\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y} \right] \qquad D\mathbb{X} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Hence if $w = f \circ \mathbb{X} = g$

$$Dg(r,\theta) = \left[\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y} \right] \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\parallel$$

$$\left[\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta}\right] = \left[\cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}, -r\sin\theta \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y}\right]$$

Thus we find, (see §11.5 # 37, I mean you can find this w/o the matrix approach)

$$\frac{\partial w}{\partial r} = \cos\theta \frac{\partial w}{\partial x} + \sin\theta \frac{\partial f}{\partial y} \longrightarrow \boxed{\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}}$$

$$\frac{\partial w}{\partial \theta} = -r\sin\theta \frac{\partial w}{\partial x} + r\cos\theta \frac{\partial w}{\partial y} \longrightarrow \boxed{\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}}$$

this is an operator eq^{n}!

$\boxed{E70}$ Let $z = f(x,y) = x^2 - 3y^2$ and let $x = uv$ & $y = u + v^2$ calculate $\partial z/\partial u$ and $\partial z/\partial v$.

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\Big[f(x(u,v), y(u,v))\Big] = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = (2x)(v) - 6y(1).$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\Big[f(x(u,v), y(u,v))\Big] = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = (2x)(u) - 6y(2v).$$

this is simple enough, you can use a tree-diagram if you like, but I've never needed them, you just identify the intermediate variables and sort-of "conserve partials". Lets see how this is done in the matrix / Jacobean formalism. We define

$$\mathbb{X}(u,v) \equiv (x(u,v), y(u,v)) = (uv, u+v^2).$$

Thus, notice $x_1 = u$ and $x_2 = v$ while $x = f_1$, $y = f_2$ and $\mathbb{X} = f$ to use 308,

$$D\mathbb{X} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \quad \text{while} \quad Df = \left[\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y}\right]$$

Then $z = f \circ \mathbb{X}$ so $z = z(u,v)$

$$Dz = \left[\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right] = (Df)(D\mathbb{X}) = \left[\frac{\partial f}{\partial x} \;\; \frac{\partial f}{\partial y}\right]$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \Big[\underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}}_{\frac{\partial z}{\partial u}}, \underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}}_{\frac{\partial z}{\partial v}}\Big]$$

So you see, its just the same formulas, the nice thing about the matrix is you get everything at once. Pragmatically speaking its the same calculation for particular physical problems. But, every view is another tool, it may make certain general arguments much more efficient. I mean just think about the chain rule, it gets <u>all</u> the other chain rules.

~ End of Theoretical Digression ~