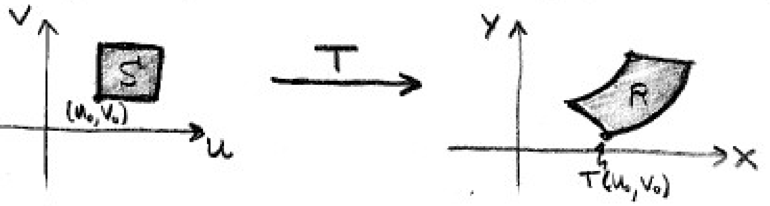
Change of Variables in Multivariate Integration

I'll sketch the proof that leads to the Th^{m}'s on the next page. You can see Colley §5.5 for a more complete argument or Stewart. As usual the correct proof is most likely to be found in ma 426. Also we stick to two dimensions until the Th^{m}'s.

Coordinate Change

Usually we assign the Cartesian Coordinates (x,y) to the plane $\mathbb{R}^2$. However we can change the coordinates to (u,v). Its helpful to consider two planes, the (x,y)-plane and the (u,v)-plane, the coordinate change map T takes (u,v) to $T(u,v) = (x(u,v), y(u,v))$



Here $T : S \subset \mathbb{R}^2 \longrightarrow R \subset \mathbb{R}^2$. We insist that T be invertible, except possibly on the boundaries, this means the eq^{n}'s relating x,y and u,v can be solved for either x,y or u,v. From our discussion of differentiability we learned T can be approximated by the function,

$$\boxed{h(u,v) = T(u_0,v_0) + DT(u_0,v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}}$$

where $DT(u_0,v_0)$ is the 2×2 Jacobean Matrix. If we consider a little parallelogram at (u_0,v_0) then transports it to $T(u_0,v_0)$ and $DT(u_0,v_0)$ distorts it into a modified parallelogram. Moreover the following proposition tells us how the areas of the parallelograms are related.

Prop: (5.1 of Colley): Let $T(u,v) = A\begin{bmatrix} u \\ v \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A \neq 0$, then if D^* is a parallelogram then $T(D^*) = D$ is also a parallelogram and $\boxed{(\text{Area of } D) = |\det A| \cdot (\text{Area of } D^*)}$

While T itself is not usually linear, its best linear approx h is and close to some particular point (u_0,v_0) it's matrix is $DT(u_0,v_0)$. Thus for a tiny rectangle $(\Delta u\, \Delta v)$ we find

$$\boxed{\Delta u\, \Delta v = \det(DT(u_0,v_0))\, \Delta x\, \Delta y}$$

then in the limit these become differentials, and also as we sum over S and $T(S) = R$ we find the theorems that follow,

How to change coordinates on a double integral

I sketched the general idea, now let me give the practical formulas so we can apply these to specific problems.

Def^{n}/ The Jacobean of the transformation $T(u,v) = (x(u,v), y(u,v))$ is,

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u y_v - x_v y_u$$

The "Jacobean" is the determinant of what I called the "Jacobean matrix",

E104 Consider polar coordinates: $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Lets calculate the JACOBEAN, note $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

Th^{m} (Changing variables in double integrals): Suppose $T : S \rightarrow R$ is a differentiable mapping that is mostly invertible (except possibly on the boundary) from TYPE I OR II region S to TYPE I OR II region R and suppose that f is a continuous function whose domain includes R,

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the $|\cdot|$ on the Jacobean are absolute value bars.

E105 Lets apply this Th^{m} to POLAR COORDINATES, suppose f is continuous etc...

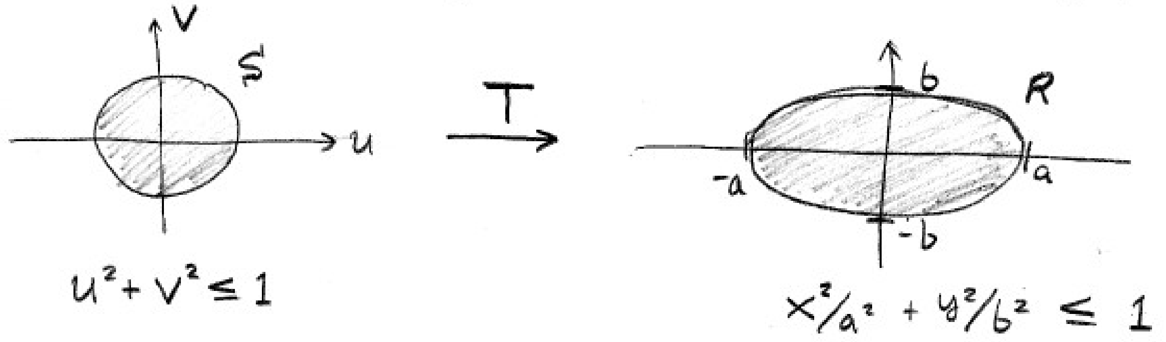
$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

E106 Using **E105** calculate the area of a circle of radius A, call it R

$$\iint_R dx\,dy = \iint_S r\,dr\,d\theta = \int_0^{2\pi}\int_0^{A} r\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{2}A^2\,d\theta = \tfrac{1}{2}A^2 \cdot 2\pi = \boxed{\pi A^2}$$

Remark: this is considerably easier than the direct Cartesian calculation of area, although the same geometry makes both sol^{n}'s work. Notice "mostly invertible" is a needed qualifier since the angle θ doubles up on $\theta = 0$ and 2π given (x,y) along $\theta = 0$ should we say it corresponds to $\theta = 0$ or $\theta = 2\pi$? Fortunately a curve or two will not change double integral's result.

$\boxed{E107}$ Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $a, b > 0$.
Find the Area, use a change of coordinates that makes it a circle



we want to find T that morphs the circle in the uv-plane into our ellipse, a bit of reflection reveals

$$\begin{aligned} x &= au \\ y &= bv \end{aligned} \qquad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = u^2+v^2 \text{, see it works} \right)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = ab - 0 = ab$$

Now calculating the area is easy given $\boxed{E106}$

$$\text{Area}(R) = \iint_R dx\,dy = \iint_S |ab|\,du\,dv = ab\iint_S du\,dv = \boxed{\pi ab = \text{Area}(R)}$$

in the last step I have used that the area of the unit circle in the uv-plane has area π.

<u>Remark</u>: we can do an analogous change of variables to obtain the volume of an ellipsoid from the volume of a sphere.

- Next we consider an example where it is not obvious how the coordinate change map's domain should be defined. The form of $T(u,v)$ will be suggested by the integrand rather than R as in $\boxed{E107}$. Please understand E107 is a novelty, on the other hand $\boxed{E108}$ is more typical. The main use of the coordinate change Th^{m}'s is certainly the standard non-Cartesian systems (Polars, Cylindricals, Sphericals).

E108 Evaluate the integral by performing an appropriate coordinate change,

$$I = \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

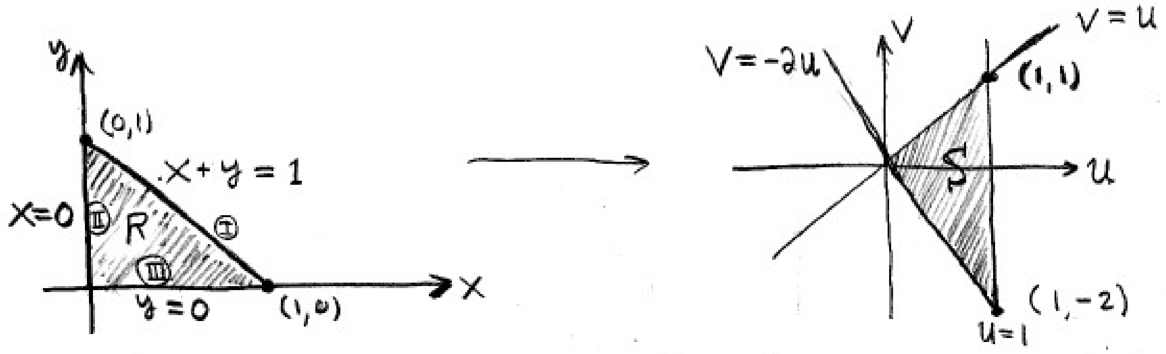
This suggests we choose $u = x+y$ and $v = y-2x$. Solving for x, y yields $x = \frac{u}{3} - \frac{v}{3}$ and $y = \frac{2u}{3} + \frac{v}{3}$. Thus

$$T(u,v) = \left(\tfrac{1}{3}(u-v),\ \tfrac{1}{3}(2u+v)\right)$$

if $T: S \to R$ then what is S in this case? We are interested in R that is indicated by the integral I, namely

$$R: \left\{ \begin{array}{c} 0 \leq y \leq 1-x \\ 0 \leq x \leq 1 \end{array} \right\}$$

WHAT IS S HERE?



To figure out the boundaries in the uv-triangle we are guided by the knowledge that for a simple linear T as we have here triangles go to triangles, vertices to vertices.

I: $x + y = \frac{1}{3}u - \cancel{\frac{1}{3}v} + \frac{2}{3}u + \cancel{\frac{1}{3}v} = u = 1$

II: $0 = x = \frac{1}{3}u - \frac{1}{3}v \Rightarrow u = v$

III.: $0 = y = \frac{2}{3}u + \frac{1}{3}v \Rightarrow v = -2u$

And we may explicitly describe S now,

$$S: \left\{ \begin{array}{c} -2u \leq v \leq u \\ 0 \leq u \leq 1 \end{array} \right\}$$

Notice then, for $x = \frac{1}{3}(u-v)$, $y = \frac{1}{3}(2u+v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) - \left(\tfrac{-1}{3}\right)\left(\tfrac{2}{3}\right) = \frac{3}{9} = \frac{1}{3}.$$

$\boxed{\text{E108 Continued}}$ Apply what we've learned.

$$I = \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$$= \int_0^1 \int_{-2u}^{u} \sqrt{u}\; v^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

$$= \int_0^1 \int_{-2u}^{u} \tfrac{1}{3} v^2 \sqrt{u}\,dv\,du$$

$$= \int_0^1 \frac{\sqrt{u}}{9}\, v^3 \Big|_{-2u}^{u} du$$

$$= \int_0^1 \frac{\sqrt{u}}{9}\left(u^3 - (-2u)^3\right) du$$

$$= \int_0^1 u^{3+1/2}\,du$$

$$= \frac{2}{9} u^{9/2}\Big|_0^1$$

$$= \boxed{2/9}$$

- note, this example borrowed from Thomas' Calculus 10^{th} Ed. pg. 1040.
- We move on to triple integrals.

Def^n/ Let $T(u,v,w) = (x(u,v,w),\, y(u,v,w),\, z(u,v,w))$ be a differentiable function from $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, the JACOBEAN of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$\boxed{\text{E109}}$ Cylindrical Coordinates : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & y_r & z_r \\ x_\theta & y_\theta & z_\theta \\ x_z & y_z & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (r\cos^2\theta + r\sin^2\theta)\,1 = \boxed{r}$$

here $r^2 = x^2 + y^2$.

E110 Spherical Coordinates:

$$x = \rho\cos\theta\sin\varphi \qquad 0 \le \theta \le 2\pi$$
$$y = \rho\sin\theta\sin\varphi \qquad 0 \le \varphi \le \pi$$
$$z = \rho\cos\varphi \qquad \rho^2 = x^2+y^2+z^2 = r^2+z^2.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\theta & y_\theta & z_\theta \\ x_\varphi & y_\varphi & z_\varphi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\rho\sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi & -\rho\sin\varphi \end{vmatrix}$$

$$= \cos\theta\sin\varphi\,(\rho\cos\theta\sin\varphi)(-\rho\sin\varphi) - \sin\theta\sin\varphi\,[\rho^2\sin\theta\sin^2\varphi] + \cos\varphi\,[-\rho\,\underline{\sin^2\theta}\sin\varphi\cos\varphi - \rho\,\underline{\cos^2\theta}\sin\varphi\cos\varphi]$$

$$= -\rho^2(\cos^2\theta\sin^3\varphi + \sin^2\theta\sin^3\varphi + \cos^2\varphi\sin\varphi)$$

$$= -\rho^2(\sin^3\varphi + \sin\varphi\cos^2\varphi)$$

$$= -\rho^2\sin\varphi\,(\sin^2\varphi + \cos^2\varphi)$$

$$= \boxed{-\rho^2\sin\varphi}$$

(notice that $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin\varphi$ thanks to determinants skew symmetry)

E111 Volume of a sphere of radius A, call it S

$$\text{vol}(S) = \iiint_S dV$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^A \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Using the triple integral coordinate change Th$^{\text{m}}$ which is stated on next page,

$$= \left(\int_0^\pi \sin\varphi\,d\varphi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^A \rho^2\,d\rho\right)$$

$$= 2\cdot 2\pi\cdot \frac{1}{3}A^3$$

$$= \boxed{\frac{4}{3}\pi A^3}$$

Th^m (Coordinate Change in Triple Integrals): Suppose that we have a differentiable, mostly invertible $T(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w))$ and a function f which is continuous on S, where $T(S) = R$.

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(T(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

- notice that $f(T(u,v,w))$ is simply notation for saying that f is to be written in terms of u,v,w as indicated by the formulas for $x(u,v,w)$, $y(u,v,w)$, $z(u,v,w)$.

$\boxed{E112}$ Sphericals.

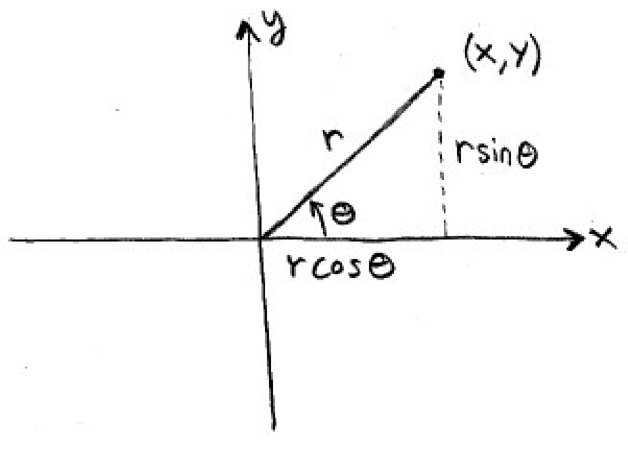
$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\cos\theta\sin\varphi,\ \rho\sin\theta\sin\varphi,\ \rho\cos\varphi)\cdot \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

we'll discuss how to see S from R in more detail later on.

$\boxed{E113}$ Cylindricals.

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(r\cos\theta,\ r\sin\theta,\ z)\, r\,dr\,d\theta\,dz$$

- We pause now to go back and cover §9.6 on the geometry of spherical and cylindrical coordinates. To begin we recall the set-up for POLAR COORDINATES,



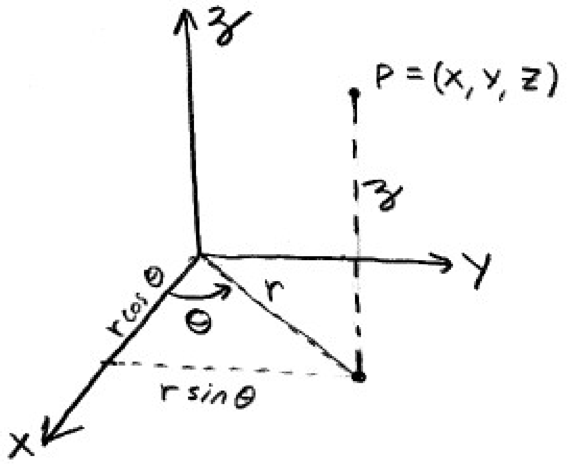
Useful Facts.

$x = r\cos\theta$ $y = r\sin\theta$	$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$
$r^2 = x^2 + y^2$ $\tan\theta = y/x$	
$0 \le \theta \le 2\pi$ $0 \le r$	

Notice that the sign of x & y is automatically encoded by the sign of $\cos\theta$ & $\sin\theta$. Also notice that θ is ambiguous at the origin and θ is double valued along $y=0,\ x>0$.

Cylindrical Coordinates:

Given a point $P \in \mathbb{R}^3$ we can describe the point P by the Polar Coordinates of $\Pi_{xy}(P)$ its projection into the xy-plane and the z-coordinate of P. As usual we relate them to the standard CARTESIAN COORDINATES,



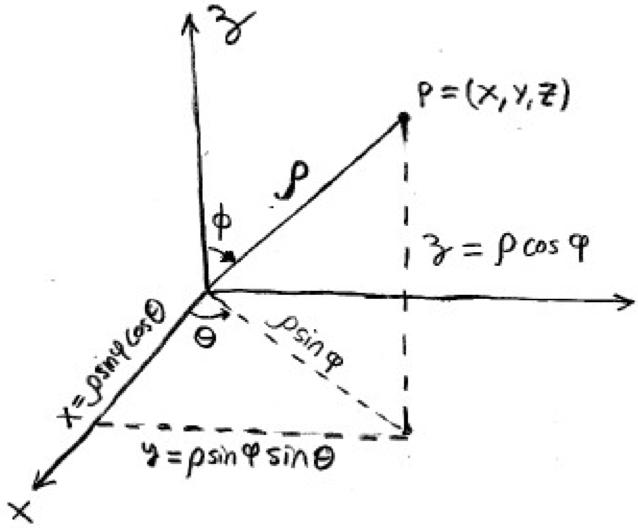
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

here the cylindrical coordinates (r, θ, z) are req^{d} to satisfy

$$r \geq 0, \quad 0 \leq \theta \leq 2\pi$$
$$r^2 = x^2 + y^2$$

Spherical Coordinates

Given a point $P \in \mathbb{R}^3$ we describe the location of P by its distance from the origin "rho" ρ and the polar angle θ plus the azimuthial angle $\phi = \varphi$ (I use both, sorry).



$$\boxed{\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}}$$

where you can easily prove

$$\rho^2 = x^2 + y^2 + z^2$$

and we require (define)

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$$
$$\rho \geq 0$$

OTHER CONVENTIONS, BEWARE: (see 383 for more)

Remark: I'm not particularly enamored with there math-conventions. To my taste the physics conventions of switching $\theta \leftrightarrow \varphi$ are nice,

$$x = r\sin\theta\cos\varphi \qquad x = s\cos\varphi$$
$$y = r\sin\theta\sin\varphi \qquad y = s\sin\varphi$$
$$z = r\cos\theta \qquad z = z$$

here

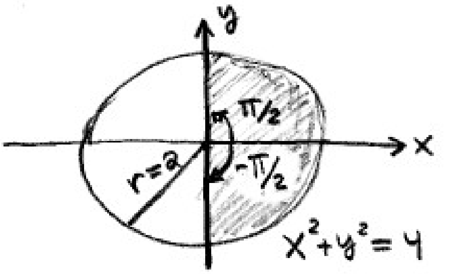
$$r^2 = x^2 + y^2 + z^2 = s^2 + z^2$$

I will use math conventions unless otherwise explicitly stated.

Double Integrals using Polars (§12.4)

We now have all the theory we need to justify these calculations.

E114 Let $R = \{ (x,y) \mid x^2+y^2 \leq 4,\ x \geq 0 \}$. Convert this region to Polars and integrate $f(x,y) = \sqrt{4-x^2-y^2}$.



$$S = \{ (r,\theta) \mid 0 \leq r \leq 2,\ \underbrace{-\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}}_{} \}$$

ok, so technically we ought to instead use $\theta \in [0, \pi/2] \cup [\frac{3\pi}{2}, 2\pi]$. but $[-\pi/2, \pi/2]$ gives the same results. I think you'll find most folks are not super picky about the domain of θ.

$$\iint_R \sqrt{4-x^2-y^2}\, dA = \int_0^2 \int_{-\pi/2}^{\pi/2} \sqrt{4-r^2\cos^2\theta - r^2\sin^2\theta}\, \underline{r\,d\theta\,dr}$$

$dA = r\,d\theta\,dr$

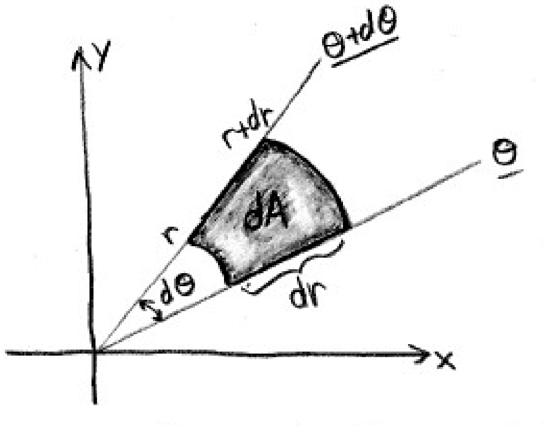
$$= \int_0^2 \int_{-\pi/2}^{\pi/2} r\sqrt{4-r^2}\, d\theta\, dr$$

$$= \int_0^2 \pi r \sqrt{4-r^2}\, dr$$

$$= \frac{-\pi}{2}\frac{2}{3}(4-r^2)^{3/2}\Big|_0^2$$

$$= \frac{-\pi}{3}\left[0 - (2^2)^{3/2}\right] = \boxed{\frac{8\pi}{3}}$$

__Remark__: We used $dA = r\,d\theta\,dr$, this was derived back in **E104–105** on 344 let me give the infinitesimal argument for this, some of you may find this more logically appealing then the general Jacobean theory. A little "__polar__ __rectangle__".



Recall from middle school, area of a wedge of radius a of angle $\Delta\theta$ is $\frac{1}{2}a^2\Delta\theta$. You are familiar with $\Delta\theta = 2\pi$ then the wedge is the whole circle and we get πa^2.

- __Using 1st order formalism__

$$dA = \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2}r^2 d\theta = \frac{1}{2}(r^2 + 2r\,dr + \underbrace{\cancel{(dr)^2}}_{\to 0} - r^2)d\theta = \boxed{r\,dr\,d\theta = dA}$$

(2nd order small).

<u>Remark Continued:</u> Another method to derive $dA = r\,dr\,d\theta$ is to use "differential forms". You may do a Bonus project on differential forms if you wish (not a req^{d} topic). In the differential forms set-up we put "wedges" between the dx and dy so $dA = dx \wedge dy = -dy \wedge dx$, the wedge product is antisymmetric. You can talk to me outside class if you'd like to know more, or see my ma 430 notes which are also posted online. For now I give the derivation as a short advertisement for differential forms,

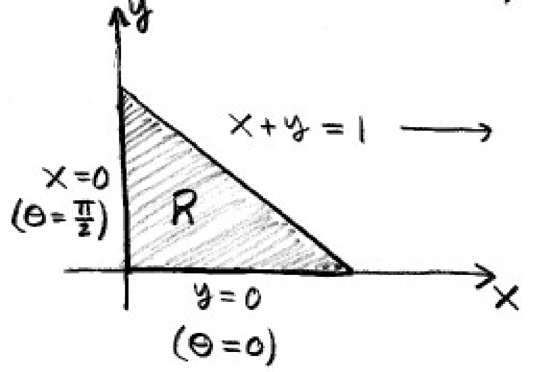
$$\begin{aligned}
dA &= dx \wedge dy \\
&= d(r\cos\theta) \wedge d(r\sin\theta) \\
&= [\cos\theta\, dr - r\sin\theta\, d\theta] \wedge [\sin\theta\, dr + r\cos\theta\, d\theta] \\
&= \sin\theta\cos\theta\, \overset{0}{\cancel{dr \wedge dr}} + r\cos^2\theta\, dr\wedge d\theta - r\sin^2\theta\, d\theta \wedge dr - r^2\sin\theta\cos\theta\, \overset{0}{\cancel{d\theta\wedge d\theta}} \\
&= r\cos^2\theta\, dr\wedge d\theta + r\sin^2\theta\, dr \wedge d\theta \\
&= \boxed{r\,dr\wedge d\theta = dA}
\end{aligned}$$

here $dA = -r\,d\theta \wedge dr$ also, in contrast to our usual $dA = r\,dr\,d\theta = r\,d\theta\,dr$ the minus sign encodes the "orientation". The "$\wedge$" automatically ignored uninteresting terms.

$\boxed{\text{E115}}$ Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ find $\displaystyle\iint_R f(x,y)\,dA$ where R is the region in xy-plane with $1 \le r \le 2$ and $0 \le \theta \le \pi/3$.

$$\begin{aligned}
\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA &= \int_0^{\pi/3}\int_1^2 \frac{1}{r}\, r\,dr\,d\theta \\
&= \left(\int_0^{\pi/3} d\theta\right)\left(\int_1^2 dr\right) \\
&= \boxed{\frac{2\pi}{3}}
\end{aligned}$$

$\boxed{E116}$ Find the area of the triangle bounded by $x=0$, $y=0$ and $x+y=1$ using POLAR COORDINATES. Its fairly easy to see that $0 \le \theta \le \pi/2$ on R, however bounding r requires thought,



$$x+y=1 \longrightarrow r\cos\theta + r\sin\theta = 1$$
$$r(\cos\theta+\sin\theta) = 1$$
$$r = \frac{1}{\cos\theta+\sin\theta}$$
$$\Rightarrow \ 0 \le r \le \frac{1}{\cos\theta+\sin\theta}$$

this is a less trivial polar region, we <u>must</u> put the integration over dr first since its bounds are θ-dependent.

$$\text{Area}(R) = \int_0^{\pi/2}\int_0^{\frac{1}{\cos\theta+\sin\theta}} r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \frac{1}{2}\left(r^2\Big|_0^{1/[\cos\theta+\sin\theta]}\right)d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}\frac{d\theta}{(\cos\theta+\sin\theta)^2} \quad : \quad \underline{\sin\theta+\cos\theta = \sqrt{2}\sin(\theta+\pi/4)}\ (*)$$
$$= \frac{1}{2}\int_0^{\pi/2}\frac{1}{2}\csc^2(\theta+\pi/4)\,d\theta$$
$$= \frac{-1}{4}\cot(\theta+\pi/4)\Big|_0^{\pi/2}$$
$$= \frac{-1}{4}\left(\cot(3\pi/4) - \cot(\pi/4)\right) =$$
$$= \frac{-1}{4}(-1-1) = \boxed{1/2}$$

$(*)$ $\sin(\theta+\pi/4) = \sin\theta\cos\pi/4 + \sin\pi/4\cos\theta = \frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$
thus we find $\sin\theta + \cos\theta = \sqrt{2}\sin(\theta+\pi/4)$.

<u>Remark:</u> this is a horrible method to find the area of a triangle. But, it illustrates a general principle which is that coordinates should be chosen to fit the problem. Obviously this problem suggests Cartesians instead.

E117 Evaluate $\iiint_E (x^3+xy^2)\,dV$ where E is the solid in the 1st octant which lies beneath the paraboloid $z = 1-x^2-y^2$. This problem suggests a cylindrical approach, notice

$$x^3 + xy^2 = x(x^2+y^2) = xr = r^2\cos\theta$$
$$z = 1-x^2-y^2 = 1-r^2$$

Also, the "first octant" is bounded by $x=0$, $y=0$ and $z=0$. On $z=0$ we find the intersection of $z = 1-r^2 = 0 \Rightarrow r^2=1$ ah ha its just the unit circle. We see

$$E:\quad 0 \le \theta \le 2\pi,\quad 0 \le r \le 1,\quad 0 \le z \le 1-r^2.$$

So calculate, recall $dV = r\,dr\,d\theta\,dz$,

$$\iiint_E (x^3+xy^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} (r^2\cos\theta)(r\,dz\,dr\,d\theta)$$

: bounds give the order of the $dz, dr, d\theta$ & vice-versa.

$$= \int_0^{2\pi}\cos\theta\,d\theta\int_0^1 r^3(1-r^2)\,dr$$

$$= \boxed{0}\,,\quad \text{since } \int_0^{2\pi}\cos\theta\,d\theta = \sin(2\pi)-\sin(0) = 0.$$

Remark: $f(x,y) = x(x^2+y^2)$ is half positive & half negative on the projection of E onto the xy-plane, $\Pi_{xy}(E) = \{(x,y,0) \mid x^2+y^2 \le 1\}$. We could have seen this one coming.

E118 Consider $f(x,y,z) = \left(e^{\sqrt{x^2+y^2+z^2}}\right)\frac{1}{x^2+y^2+z^2}$. Find $\iiint_E f\,dV$ where $E = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 9\} \Rightarrow 1 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$.

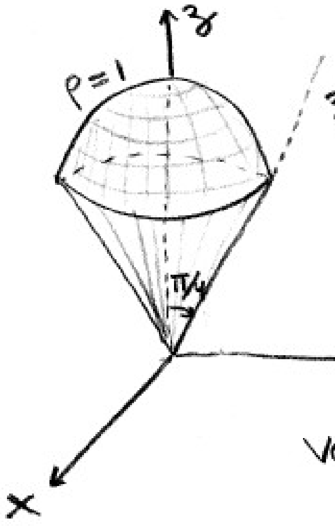
$$\iiint_E \frac{1}{x^2+y^2+z^2}\,e^{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_1^3 \frac{1}{\rho^2}e^{\rho}\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$= \left(\int_0^{\pi}\sin\varphi\,d\varphi\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_1^3 e^{\rho}\,d\rho\right)$$

$$= (-\cos\pi + \cos 0)(2\pi)(e^3-e^1)$$

$$= \boxed{4\pi(e^3-e)}$$

E119 Find the volume and centroid of the region E which is above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$. We assume that E has a constant density δ.



$\underbrace{z = \sqrt{y^2} = y}$ makes angle of $\pi/4$ with z-axis.

In spherical coordinates the region E is easy to describe,

$$0 \le \theta \le 2\pi$$
$$0 \le \rho \le 1$$
$$0 \le \varphi \le \pi/4.$$

$$\begin{aligned}
\mathrm{vol}(E) &= \int_0^{\pi/4}\int_0^1\int_0^{2\pi} \rho^2 \sin\varphi \, d\theta\, d\rho\, d\varphi \\
&= \int_0^1 \rho^2 d\rho \int_0^{\pi/4} \sin\varphi\, d\varphi \int_0^{2\pi} d\theta \\
&= \left(\tfrac{1}{3}\right)\left(-\cos(\pi/4) + 1\right) 2\pi \\
&= \boxed{\tfrac{2\pi}{3}\left(1 - \tfrac{1}{\sqrt{2}}\right)}
\end{aligned}$$

Now since δ = constant the total mass is $M = \mathrm{vol}(E)\cdot\delta = \frac{2\pi\delta}{3}(1 - 1/\sqrt{2})$

The "centroid" is the center of mass it is defined to be $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ where $\bar{x} = M_{yz}/M$, $\bar{y} = M_{zx}/M$ and $\bar{z} = M_{xy}/M$, and M_{yz}, M_{zx}, M_{xy} are the moments about the coordinate planes,

$$M_{yz} \equiv \iiint_E x\delta\, dV = 0$$
$$M_{zx} \equiv \iiint_E y\delta\, dV = 0$$

by the symmetry of the object. Calculate it, if you don't believe me.

$$\begin{aligned}
M_{xy} = \iiint_E \delta z\, dV &= \int_0^{2\pi}\int_0^1\int_0^{\pi/4} (\delta\rho\cos\varphi)(\rho^2 \sin\varphi\, d\varphi\, d\rho\, d\theta) \\
&= (2\pi\delta)\left(\rho^4/4 \Big|_0^1\right)\left(\tfrac{1}{2}\sin^2\varphi \Big|_0^{\pi/4}\right) \\
&= \frac{\pi\delta}{4}\left(\frac{1}{\sqrt{2}}\right)^2 = \pi\delta/8 = M_{xy}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \bar{x}, \bar{y}, \bar{z} \rangle &= \left\langle 0, 0, \frac{\pi\delta/8}{\frac{2\pi\delta}{3}(1 - 1/\sqrt{2})} \right\rangle \\
&= \boxed{\left\langle 0, 0, \frac{3}{16(1 - 1/\sqrt{2})} \right\rangle}
\end{aligned}$$

E120 Find the Kinetic Energy of a ball with radius R and mass m that spins with angular velocity ω and moves linearly with speed V. It is known that $KE_{net} = KE_{rot} + KE_{trans}$ where $KE_{trans} = \frac{1}{2}mV^2$ whereas $KE_{rot} = \frac{1}{2}I\omega^2$ and I = moment of intertia, lets say about the (moving) z-axis.

$$I_z = \iiint_E \delta(x,y,z)(x^2+y^2)\,dV \qquad \delta = \text{constant} = \frac{m}{\frac{4}{3}\pi R^3} = \text{density}.$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{3m}{4\pi R^3}\left[(\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2\right]\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \frac{3m}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^4\sin^3\varphi\,d\rho\,d\varphi\,d\theta \qquad : \int \sin^3\varphi\,d\varphi = \int (1-\cos^2\varphi)\sin\varphi\,d\varphi$$

$$= \frac{3m}{4\pi R^3}\left(d\theta\Big|_0^{2\pi}\right)\left(\frac{1}{3}\cos^3\varphi - \cos\varphi\Big|_0^{\pi}\right)\left(\frac{\rho^5}{5}\Big|_0^{R}\right)$$

$$= \frac{3m}{4\pi R^3}(2\pi)\left(\frac{-1}{3} + 1 - \frac{1}{3} + \cos 0\right)\left(\frac{1}{5}R^5\right)$$

$$= \frac{\cancel{3}m}{\cancel{4\pi} R^3}(2\cancel{\pi})\left(\frac{\cancel{4}}{\cancel{3}}\right)\left(\frac{1}{5}R^5\right)$$

$$= \boxed{\frac{2}{5}mR^2 = I_{sphere}}$$

Then we can calculate (don't worry this isn't physics you're not expected to know all this by heart)

$$KE_{total} = \frac{1}{2}mV^2 + \frac{1}{5}mR^2\omega^2$$

If the ball is rolling then $\omega = V/R$ (no slipping)

$$KE_{total} = \frac{1}{2}mR^2\omega^2 + \frac{1}{5}mR^2\omega^2 = \frac{7}{10}mR^2\omega^2$$

<u>Remark</u>: This type of example is one of my main objections to the standard math notation of $\rho = x^2+y^2+z^2$. I would like to use ρ for density, your text does it anyway.

<u>Remark</u>: we'll skip §12.6 on surface area for now, later on we'll return to those problems as we cover §13.6 on surface integrals.

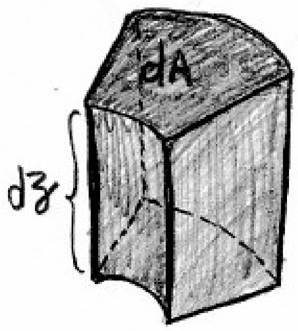
Geometry of Volume Elements (§12.8)

The Jacobean gives us the volume element in other coordinate systems if $T(u,v,w) = \langle x, y, z\rangle$ where x, y, z are fncts of u, v, w. then the infinitesimal volume dV

$$\boxed{dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

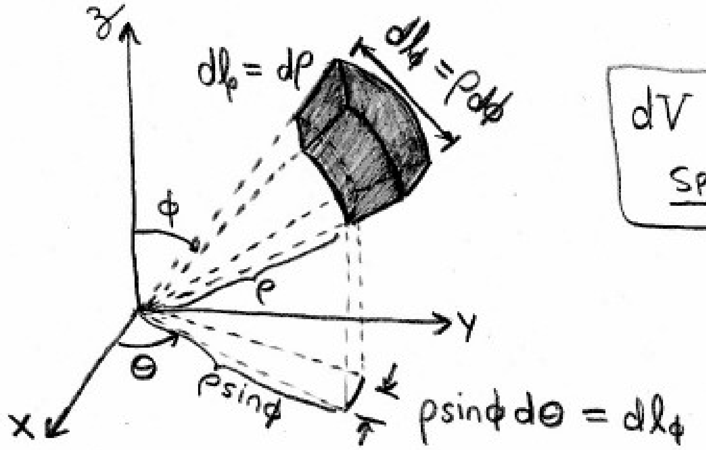
we used this general theory to derive $dV = r\,dr\,d\theta\,dz$ for cylindricals and $dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$. You might wonder if there is an explicit geometric derivation of these volume elements. The answer is yes, I'll show you how, well I'll try

$dA = r\,dr\,d\theta$ (the POLAR rectangle)



$$\boxed{dV = r\,dr\,d\theta\,dz}$$

CYLINDRICAL VOLUME ELEMENT



$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

SPHERICAL VOLUME ELEMENT

Remark: we can also use differential forms to derive these things. Again I mention this is not a req^d topic.

$$\begin{aligned} dx\wedge dy\wedge dz &= d(r\cos\theta)\wedge d(r\sin\theta)\wedge dz \\ &= \boxed{r\,dr\wedge d\theta\wedge dz = vol_3} \end{aligned}$$

where I have used what I already calculated on 352.

(Optional not req^d): Volume Element in Spherical Coordinates

Calculate as before, ask me if you'd like to know more about what the calculation below means.

$$\begin{aligned}
vol_3 &= dx \wedge dy \wedge dz \\
&= d(\rho \sin\phi \cos\theta) \wedge d(\rho \sin\phi \sin\theta) \wedge d(\rho \cos\phi) \\
&= [\sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta] \wedge \\
&\quad \wedge [\sin\phi\sin\theta\, d\rho + \rho\cos\phi\sin\theta\, d\phi + \rho\sin\phi\cos\theta\, d\theta] \wedge \\
&\quad \wedge [\cos\phi\, d\rho - \rho\sin\phi\, d\phi] \\
&= [\sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta] \wedge \\
&\quad \wedge [-\rho\sin^2\phi\sin\theta\, d\rho\wedge d\phi + \rho\cos^2\phi\sin\theta\, d\phi\wedge d\rho + \\
&\quad + \rho^2\sin\phi\cos\phi\cos\theta\, d\theta\wedge d\rho - \rho^2\sin^2\phi\cos\theta\, d\theta\wedge d\phi] \\
&= -\rho^2\sin^3\phi\cos^2\theta\, d\rho\wedge d\theta\wedge d\phi + \rho^2\cos^2\phi\sin\phi\cos^2\theta\, d\phi\wedge d\theta\wedge d\rho \\
&\quad + \rho^2\sin^3\phi\sin^2\theta\, d\theta\wedge d\rho\wedge d\phi - \rho^2\sin\phi\cos^2\phi\sin^2\theta\, d\theta\wedge d\phi\wedge d\rho \\
&= [-\rho^2\sin^3\phi\cos^2\theta - \rho^2\cos^2\phi\sin\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\sin^2\theta]\, d\rho\wedge d\theta\wedge d\phi \\
&= \rho^2\sin\phi\,[\sin^2\phi\cos^2\theta + \cos^2\theta\cos^2\phi + \sin^2\phi\sin^2\theta + \sin^2\theta\cos^2\phi]\, d\rho\wedge d\phi\wedge d\theta \\
&= \rho^2\sin\phi\,[\cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta(\sin^2\phi + \cos^2\phi)]\, d\rho\wedge d\phi\wedge d\theta \\
&= \boxed{\rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta = vol_3}
\end{aligned}$$

In fact the wedge product is just another way to calculate the determinant. You could even define the determinant implicitly by the following formula,

$$\boxed{Ae_1 \wedge Ae_2 \wedge \cdots \wedge Ae_n \equiv \det(A)\, e_1\wedge e_2 \wedge\cdots\wedge e_n}$$

For example $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$ gives $Ae_1 = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\c\end{bmatrix} = ae_1 + ce_2$ and $Ae_2 = A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}b\\d\end{bmatrix} = be_1 + de_2$ thus

$$\begin{aligned}
Ae_1 \wedge Ae_2 &= (ae_1 + ce_2)\wedge(be_1 + de_2) \\
&= ab\, e_1\wedge e_1 + ad\, e_1\wedge e_2 + cb\, e_2\wedge e_1 + cd\, e_2\wedge e_2 \\
&= (ad - cb)\, e_1\wedge e_2 \equiv \det(A)\, e_1\wedge e_2 \Rightarrow \boxed{\det A = ad - bc}
\end{aligned}$$

that is of course the usual formula for a 2x2 determinant.

A COMMENT ON n-dim'l integration

Let $\mathbb{R}^n$ have CARTESIAN COORDINATES $(x_1, x_2, .., x_n)$ and suppose $T(x_1, x_2, .., x_n) = (y_1, y_2, ..., y_n)$ where y_i is a function of $x_1, .., x_n$ for each i and we suppose T has DT invertible over the domain of integration below,

$$\iint_S \cdots \int f(x_1, x_2, .., x_n)\, d^n x = \iint_{T(S)} \cdots \int f(y_1, y_2, ..., y_n) \left| \det(DT) \right| d^n y$$

where $d^n x \equiv dx_1 dx_2 \cdots dx_n$ and $d^n y = dy_1 dy_2 \cdots dy_n$. The meaning of the n-fold integration should be an easy generalization of the cases we've already considered $n=2$ and 3.

$\boxed{E121}$ The HYPERSPHERE : $(x, y, z, w) \in \mathbb{R}^4$ such that $x^2+y^2+z^2+w^2 \leq R^2$, Generalized Spherical Coordinates are

$$\begin{aligned} x &= r\cos\theta \sin\phi \sin\psi \\ y &= r\sin\theta \sin\phi \sin\psi \\ z &= r\cos\phi \sin\psi \\ w &= r\cos\psi \end{aligned} \qquad \begin{pmatrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi, \psi \leq \pi \\ 0 \leq r \end{pmatrix}$$

you can check that $x^2+y^2+z^2+w^2 = r^2$. Then its a long but straightforward calculation,

$$\left| \frac{\partial(x, y, z, w)}{\partial(r, \theta, \phi, \psi)} \right| = r^3 \sin^2\psi \sin\phi$$

And consequently if we integrate $dx\,dy\,dz\,dw$ and change to sphericals we'll find,

$$\text{vol}_4(\text{HYPERSPHERE}) = \int_0^R \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} r^3 \sin^2\psi \sin\phi \, d\psi\, d\phi\, d\theta\, dr = \frac{\pi^2 R^4}{2}.$$

(if you like this sort of digression take a look at my ma 430 notes.)