Vector Fields

A vector field is an assigment of a vector at each point in space. Usually we take the vector field to be static, but you could imagine that the assigment itself had a time dependence.

> Def^{n}/ Let D be a subset of $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x,y) in D a 2-dim'l vector $F(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$.

we call F_1 and F_2 the component functions of F.

> Def^{n}/ Let $E \subseteq \mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x,y,z) in E a 3-dim'l vector
> $$F(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$$

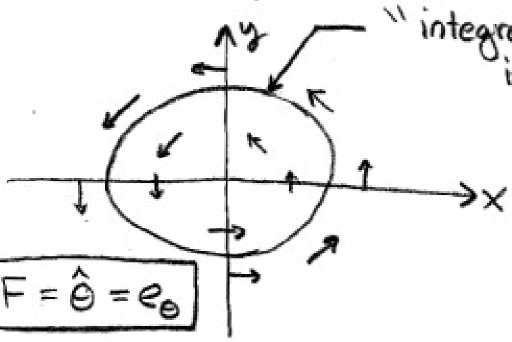
The text likes to use the notation P,Q or P,Q,R for the component functions so $F = \langle P, Q\rangle$ in 2-dimensions or $F = \langle P,Q,R\rangle$ in $\mathbb{R}^3$.

- Each vector field is made of several scalar-valued functions. We could say a vector field is a vector-valued function of $\mathbb{R}^n$. There are many interesting vector fields,
  - $\vec{E}$ the Electric Field
  - $\vec{B}$ the Magnetic Field
  - $\vec{F}$ a force field
  - $\vec{F} = -\dfrac{GmM}{r^2}\hat{r}$
  - $\vec{V}$ the velocity field of some liquid
  - $\vec{F} = -\nabla U$ a <u>conservative</u> vector field, U is a scalar function we call the <u>potential energy</u>.

<u>Remark</u>: We have already played with vector fields back when we consider the operation of gradient. The object ∇f is a vector field. It turns out that not all vector fields have the form $F = \nabla f$. When $\exists f$ such that $F = \nabla f$ we say F is conservative.

$\boxed{\text{E122}}$ Consider $x^2+y^2=1$. Diff. implicitly $\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$ in calc II. we plotted $\frac{dy}{dx}$ at each (x,y) to describe the so-called <u>direction-field</u>. This is another important example of vector fields.

or the flow line of F is another perspective see §13.1 #35



$\boxed{F = \hat{\theta} = e_\theta}$

$\underbrace{\text{"integral curve of the direction field"}}$ it is a sol$^{\underline{n}}$ of $\dfrac{dy}{dx} = \dfrac{-x}{y}$

the little arrows are derived from the formula $dy/dx = -x/y = \dfrac{\text{rise}}{\text{run}}$

its $\approx$ $F(x,y) = \langle -y, x \rangle$ although perhaps $F(x,y) = -y\hat{i} + x\hat{j}$ is better here.

- <u>flow lines of a vector field are paths whose tangents match the vector field</u>

Standard Basis for Vectors in Polar Coordinates

We wish to desribe unit vectors that point in the direction of increasing r and θ. Notice first how we could find our familiar friends $\hat{i}$ and $\hat{j}$:

$$\hat{i} = \langle 1, 0 \rangle = \nabla x \equiv \hat{x}$$
$$\hat{j} = \langle 0, 1 \rangle = \nabla y \equiv \hat{y}$$

observe that the gradient operation reveals just the vector we're looking for. Recall $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$

$$\nabla r = \frac{1}{2\sqrt{x^2+y^2}} \langle 2x, 2y \rangle = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j}$$

$$\nabla\theta = \nabla\left(\tan^{-1}(y/x)\right) = \frac{1}{1+(y/x)^2}\left[\nabla(y/x)\right] = \frac{1}{1+y^2/x^2}\left[\frac{-y}{x^2}\hat{i} + \frac{1}{x}\hat{j}\right]$$

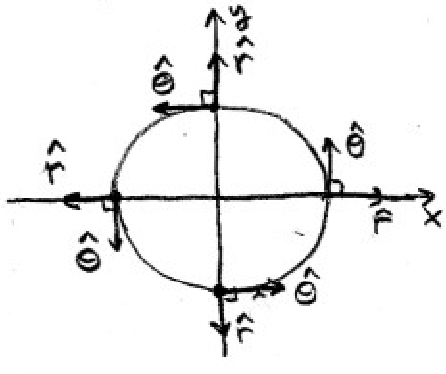
$$\Rightarrow \nabla\theta = \frac{x^2}{x^2+y^2}\left[\frac{-y}{x^2}\hat{i} + \frac{x}{x^2}\hat{j}\right] = \frac{-y}{r^2}\hat{i} + \frac{x}{r^2}\hat{j}$$

We need unit vectors. It may help to use $x = r\cos\theta$, $y = r\sin\theta$,

$$\nabla r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\nabla\theta = \frac{1}{r}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

Clearly $|\nabla r| = \cos^2\theta + \sin^2\theta = 1$, on the other hand $|\nabla\theta| = 1/r$. We find then the Standard Polar basis is

$$\boxed{\begin{aligned} \hat{r} &\equiv e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ \hat{\theta} &\equiv e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{aligned}}$$

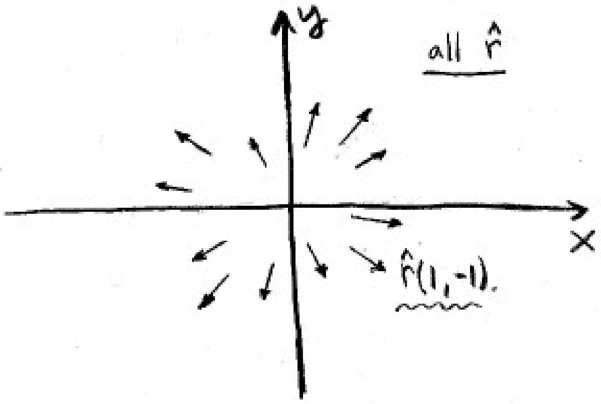


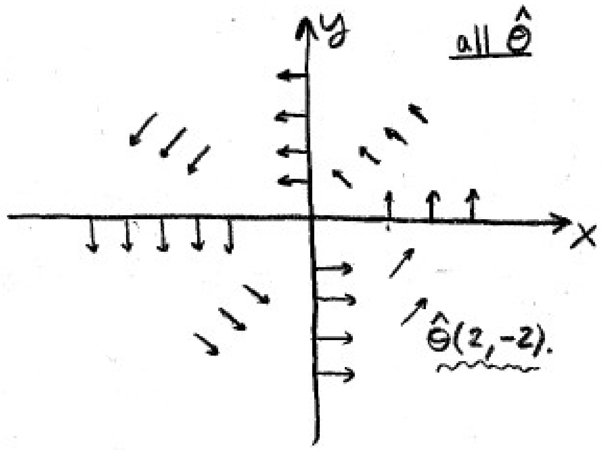
A basis is called orthogonal if the members are perpendicular. Note $\hat{r} \cdot \hat{\theta} = 0$.

<u>e_r and e_θ are vector fields!</u>

The Polar Coordinates Standard Basis : e_r, e_θ

Before I have been quite reckless exchanging points and vectors. This bothered many of you (with good cause, they're different!) It was a special quirk of CARTESIAN COORDINATES that the Standard (CARTESIAN) Basis $\hat{i}, \hat{j}$ is constant over the whole xy-plane. Now the Polar Basis or "frame" is coordinate dependent, we cannot simply move $\hat{\theta}$ or $\hat{r}$ around, their directions are given by θ.

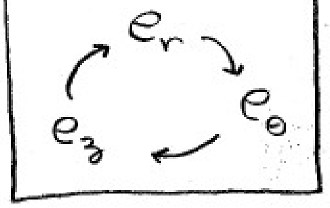




CYLINDRICAL STANDARD BASIS

Cylindrical coordinates are right-handed since $e_z = e_r \times e_\theta$

$$e_z \stackrel{?}{=} e_r \times e_\theta$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$
$$= (\cos^2\theta + \sin^2\theta)\hat{k}$$
$$= \underline{\hat{k} = e_z = \hat{z} = \langle 0,0,1\rangle}$$



Cross product works the same on these as $\hat{i}, \hat{j}, \hat{k}$

We can work with vector fields of the form

$$F = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = F_r e_r + F_\theta e_\theta + F_z e_z$$

we would like to be able to use either coordinate system. Obvious question: how do the component functions F_x, F_y, F_z relate to F_r, F_θ, F_z? Its a simple matter of algebra, we tackle it next,

$$F = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = F_r e_r + F_\theta e_\theta + F_z e_z$$

Clearly the F_z is shared by both, not surprising. Now use 361 to write

$$F_x \hat{i} + F_y \hat{j} = F_r(\cos\theta \hat{i} + \sin\theta \hat{j}) + F_\theta(-\sin\theta \hat{i} + \cos\theta \hat{j})$$
$$= (F_r \cos\theta - F_\theta \sin\theta)\hat{i} + (F_r \sin\theta + F_\theta \cos\theta)\hat{j}$$

Therefore we find

$$\boxed{\begin{aligned} F_x &= F_r \cos\theta - F_\theta \sin\theta \\ F_y &= F_r \sin\theta + F_\theta \cos\theta \end{aligned}} \quad (*)$$

In matrix notation

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta F_x + \sin\theta F_y \\ -\sin\theta F_x + \cos\theta F_y \end{pmatrix}$$

Thus the inverse relations are,

$$\boxed{\begin{aligned} F_r &= \cos\theta F_x + \sin\theta F_y \\ F_\theta &= -\sin\theta F_x + \cos\theta F_y \end{aligned}} \quad (**)$$

you could derive these relations geometrically if you prefer. Or we could use $F_r = F \cdot e_r$ and $F_\theta = F \cdot e_\theta$ and that'd be quicker.

$\boxed{E123}$ Suppose $F = \langle 1, 2, 3 \rangle$ find F in cylindrical frame.

$$F_r = \cos\theta F_x + \sin\theta F_y = \cos\theta + 2\sin\theta$$
$$F_\theta = -\sin\theta F_x + \cos\theta F_y = -\sin\theta + 2\cos\theta$$
$$F_z = F_z = 3$$

Thus $\boxed{F = (\cos\theta + 2\sin\theta)e_r + (2\cos\theta - \sin\theta)e_\theta + 3e_z}$

the components are coordinate dependent even though the original F was constant, somehow these conspire to be constant. Consider $\theta = 0$ and $\theta = \pi/2$ for example,

$$F|_{\theta=0} = (e_r + 2e_\theta + 3e_z)|_{\theta=0} = \hat{i} + 2\hat{j} + 3\hat{k} = \langle 1, 2, 3 \rangle.$$
$$F|_{\theta=\pi/2} = (2e_r - e_\theta + 3\hat{k}|_{\theta=\pi/2} = 2\hat{j} - (-\hat{i}) + 3\hat{k} = \langle 1, 2, 3 \rangle.$$

SPHERICAL STANDARD BASIS

Spherical coordinates are $\rho = \sqrt{x^2+y^2+z^2}$ and θ, φ where θ is the same polar angle as we discussed in the cylindrical case. Thus $e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \frac{-y}{r}\hat{i} + \frac{x}{r}\hat{j}$ where $r = \sqrt{x^2+y^2}$ and e_ρ is easily calculated,

$$e_\rho = \frac{\nabla\rho}{|\nabla\rho|} = \frac{x}{\rho}\hat{i} + \frac{y}{\rho}\hat{j} + \frac{z}{\rho}\hat{k} = \cos\theta\sin\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\varphi\,\hat{k}$$

finally construct e_φ as $-e_\rho \times e_\theta$:

oops should have done $e_\varphi = e_\theta \times e_\rho$ but I already finished this

$$-e_\varphi = e_\rho \times e_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= (-\cos\varphi\cos\theta)\hat{i} - (\cos\varphi\sin\theta)\hat{j} + (\cos^2\theta\sin\varphi + \sin^2\theta\sin\varphi)\hat{k}$$

$$= \underline{-\cos\varphi\cos\theta\,\hat{i} - \cos\varphi\sin\theta\,\hat{j} + \sin\varphi\,\hat{k} = -e_\varphi}$$

I would hope that $e_\varphi = \frac{\nabla\varphi}{|\nabla\varphi|}$. (You can check it).

To summarize :

$$\boxed{\begin{aligned} e_\rho &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \cos\theta\sin\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\varphi\,\hat{k} \\ e_\theta &= \frac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \\ e_\varphi &= \frac{xz\hat{i} + yz\hat{j} - (x^2+y^2)\hat{k}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} = \cos\varphi\cos\theta\,\hat{i} + \cos\varphi\sin\theta\,\hat{j} - \sin\varphi\,\hat{k} \end{aligned}} \quad (*)$$

Then if $F = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta$ we have

$$\boxed{\begin{aligned} F_\rho &= F\cdot e_\rho = \cos\theta\sin\varphi\,F_x + \sin\theta\sin\varphi\,F_y + \cos\varphi\,F_z \\ F_\theta &= F\cdot e_\theta = -\sin\theta\,F_x + \cos\theta\,F_y \\ F_\varphi &= F\cdot e_\varphi = \cos\varphi\cos\theta\,F_x + \cos\varphi\sin\theta\,F_y - \sin\varphi\,F_z \end{aligned}} \quad (**)$$

$$\boxed{\begin{tikzcd} e_\rho \arrow[dr] & \\ e_\theta \arrow[u] & e_\varphi \arrow[l] \end{tikzcd} \qquad \begin{aligned} e_\rho \times e_\varphi &= e_\theta \\ e_\varphi \times e_\theta &= e_\rho \\ e_\theta \times e_\rho &= e_\varphi \end{aligned}}$$

Sphericals are Right-Handed also.

I'd like to solve (**) (364) for F_x, F_y, F_z. I'm going to use (*) (364) and the fact $F_x = F \cdot \hat{i}$, $F_y = F \cdot \hat{j}$ and $F_z = F \cdot \hat{k}$ to do this

$$\begin{aligned} F_x &= (F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta) \cdot \hat{i} \\ &= F_\rho (e_\rho \cdot \hat{i}) + F_\varphi (e_\varphi \cdot \hat{i}) + F_\theta (e_\theta \cdot \hat{i}) \\ &= \underline{\cos\theta \sin\varphi F_\rho + \cos\theta \cos\varphi F_\varphi - \sin\theta F_\theta} \end{aligned}$$

using (*) on 364

$$\begin{aligned} F_y &= F_\rho (e_\rho \cdot \hat{j}) + F_\varphi (e_\varphi \cdot \hat{j}) + F_\theta (e_\theta \cdot \hat{j}) \\ &= \underline{\sin\theta \sin\varphi F_\rho + \sin\theta \cos\varphi F_\varphi + \cos\theta F_\theta} \end{aligned}$$

using (*) on 364

$$\begin{aligned} F_z &= F_\rho (e_\rho \cdot \hat{k}) + F_\varphi (e_\varphi \cdot \hat{k}) + F_\theta (e_\theta \cdot \hat{k}) \\ &= \underline{\cos\varphi F_\rho - \sin\varphi F_\varphi} \end{aligned}$$

Thus we find

$$\boxed{\begin{aligned} F_x &= \cos\theta \sin\varphi F_\rho + \cos\theta \cos\varphi F_\varphi - \sin\theta F_\theta \\ F_y &= \sin\theta \sin\varphi F_\rho + \sin\theta \cos\varphi F_\varphi + \cos\theta F_\theta \\ F_z &= \cos\varphi F_\rho - \sin\varphi F_\varphi \end{aligned}} \quad (*)$$

<u>Remark</u>: the reason I've gone to the trouble of calculating these relations is that it is a good warm-up to calculating the explicit formulas for gradient in other coordinates. Also it will help us find useful NON-CARTESIAN formulas for the "Curl" and "Divergence". Lets address the question of gradients for now.

GRADIENT IN POLARS, CYLINDRICALS, AND SPHERICALS

Th^m/ The gradient is found to be:

(1.) $$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta \quad : \text{ for } f : \mathbb{R}^2 \to \mathbb{R}$$

(2.) $$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z \quad : \text{ for } f : \mathbb{R}^3 \to \mathbb{R}$$

(3.) $$\nabla f = \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi} e_\phi + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta} e_\theta \quad : \text{ for } f : \mathbb{R}^3 \to \mathbb{R}$$

- Notice ∇f is a vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$ here.

Pf: We begin with case (1.). Recall that $x = r\cos\theta$ and $y = r\sin\theta$ and we found $e_r = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$ & $e_\theta = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}$. We need to solve for $\hat{i}$ and $\hat{j}$ in terms of e_r & e_θ, I'll use a standard trick,

$$\begin{aligned} \hat{i} &= a e_r + b e_\theta \\ &= (\hat{i}\cdot e_r) e_r + (\hat{i}\cdot e_\theta) e_\theta \\ &= \boxed{\cos\theta\, e_r - \sin\theta\, e_\theta = \hat{i}} \end{aligned}$$

: find a, b by using dot-product.

: generally $A = (A\cdot\hat{i})\hat{i} + (A\cdot\hat{j})\hat{j}$ I'm just applying this idea to the polar <u>basis</u>.

$$\begin{aligned} \hat{j} &= (\hat{j}\cdot e_r) e_r + (\hat{j}\cdot e_\theta) e_\theta \\ &= \boxed{\sin\theta\, e_r + \cos\theta\, e_\theta = \hat{j}} \end{aligned}$$

: same idea as in $\hat{i}$.

Consider then,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \\ &= \frac{\partial f}{\partial x}(\cos\theta\, e_r - \sin\theta\, e_\theta) + \frac{\partial f}{\partial y}(\sin\theta\, e_r + \cos\theta\, e_\theta) \\ &= \left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right)e_r + \left(-\sin\theta\frac{\partial f}{\partial x} + \cos\theta\frac{\partial f}{\partial y}\right)e_\theta \\ &= e_r\left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right]f + e_\theta\frac{1}{r}\left[-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right]f \\ &= e_r\frac{\partial}{\partial r}(f) + e_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(f) \\ &= \boxed{e_r\frac{\partial f}{\partial r} + e_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} = \nabla f} \end{aligned}$$

see p. 309 we derived this during our discussion of the chain-rule.

In polar coordinates: $\nabla = e_r\frac{\partial}{\partial r} + \frac{1}{r}e_\theta\frac{\partial}{\partial\theta}$.

Proof Continued:

The proof of (2.) follows immediately from (1.) since in that case $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = e_r \frac{\partial f}{\partial r} + \frac{1}{r} e_\theta \frac{\partial f}{\partial \theta} + e_z \frac{\partial f}{\partial z}$ since $e_z = \hat{k}$. The proof of (3.) is less obvious. To begin lets rewrite $\hat{i}, \hat{j}, \hat{k}$ in the spherical basis e_ρ, e_ϕ, e_θ,

$$\hat{i} = (\hat{i}\cdot e_\rho)e_\rho + (\hat{i}\cdot e_\phi)e_\phi + (\hat{i}\cdot e_\theta)e_\theta$$
$$= \cos\theta\sin\varphi\, e_\rho + \cos\theta\cos\varphi\, e_\phi - \sin\theta\, e_\theta = \hat{i} \quad \text{I}$$

See 364 (*) to do the dot-products

$$\hat{j} = (\hat{j}\cdot e_\rho)e_\rho + (\hat{j}\cdot e_\phi)e_\phi + (\hat{j}\cdot e_\theta)e_\theta$$
$$= \sin\theta\sin\varphi\, e_\rho + \sin\theta\cos\varphi\, e_\phi + \cos\theta\, e_\theta = \hat{j} \quad \text{II}$$

$$\hat{k} = (\hat{k}\cdot e_\rho)e_\rho + (\hat{k}\cdot e_\phi)e_\phi + (\hat{k}\cdot e_\theta)e_\theta$$
$$= \cos\varphi\, e_\rho - \sin\varphi\, e_\phi = \hat{k} \quad \text{III}$$

I used (*) of 364 to evaluate the dotproducts above. Consider,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= e_\rho\left(\cos\theta\sin\varphi\frac{\partial f}{\partial x} + \sin\theta\sin\varphi\frac{\partial f}{\partial y} + \cos\varphi\frac{\partial f}{\partial z}\right)$$
$$+ e_\phi\left(\cos\theta\cos\varphi\frac{\partial f}{\partial x} + \sin\theta\cos\varphi\frac{\partial f}{\partial y} - \sin\varphi\frac{\partial f}{\partial z}\right)$$
$$+ e_\theta\left(-\sin\theta\frac{\partial f}{\partial x} + \cos\theta\frac{\partial f}{\partial y}\right)$$

: using I, II & III gathering like terms.

$$= e_\rho\left(\frac{\partial x}{\partial \rho}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial f}{\partial z}\right)$$
$$+ \frac{1}{\rho} e_\phi\left(\rho\cos\theta\cos\varphi\frac{\partial f}{\partial x} + \rho\sin\theta\cos\varphi\frac{\partial f}{\partial y} - \rho\sin\phi\frac{\partial f}{\partial z}\right)$$
$$+ \frac{1}{\rho\sin\phi} e_\theta\left(-\rho\sin\theta\sin\phi\frac{\partial f}{\partial x} + \rho\cos\theta\sin\phi\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial f}{\partial z}\right)$$

$$= e_\rho\frac{\partial f}{\partial \rho} + \frac{1}{\rho}e_\phi\left(\frac{\partial x}{\partial \phi}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial f}{\partial z}\right) + \frac{e_\theta}{\rho\sin\phi}\left(\frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial f}{\partial z}\right)$$

$$= \boxed{e_\rho\frac{\partial f}{\partial \rho} + \frac{1}{\rho}e_\phi\frac{\partial f}{\partial \phi} + \frac{1}{\rho\sin\phi}e_\theta\frac{\partial f}{\partial \theta} = \nabla f}$$

Observe in Sphericals $\nabla = e_\rho\frac{\partial}{\partial \rho} + \frac{e_\phi}{\rho}\frac{\partial}{\partial \phi} + \frac{e_\theta}{\rho\sin\phi}\frac{\partial}{\partial \theta}$. (*)

E124 Consider the function $U = \frac{-GmM}{\rho}$. Lets find the gradient of U where G is the gravitational constant, m is the mass of a planet and M is the mass of the sun which we place at $\rho = 0$ so the distance between M & m is $\rho = \sqrt{x^2+y^2+z^2}$. Lets use our Th$^{\underline{m}}$,

$$\nabla U = -GmM\left[e_\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right) + \frac{1}{\rho} e_\phi \cancel{\frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\right)}^{0} + \frac{1}{\rho \sin\phi} e_\theta \cancel{\frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\right)}^{0} \right]$$

$$\nabla U = \frac{GmM}{\rho^2} e_\rho \quad \Rightarrow \quad F = -\nabla U = \boxed{\frac{-GmM}{\rho^2} e_\rho = F}$$

this is Newton's Universal Law of Gravitation.

E125 Consider the electric potential $V = \frac{-kQ}{\rho}$ then

$$E = -\nabla V = \boxed{\frac{kQ}{\rho^2} e_\rho = E}$$ the Electric field of a static point charge Q.

to get the force on q due to Q's electric field we would multiply by q so $F = \frac{kqQ}{\rho^2} e_\rho$ and the force is attractive if $qQ < 0$ (opposite polarity) or repulsive if $qQ > 0$ (like charges) in contrast to $F_{gravity}$ which has same $1/\rho^2$ dependence but is always attractive.

E126 Suppose $x^2+y^2+z^2 = R^2$ find the normal to this level surface. Notice this is $\rho^2 = R^2$ or $\rho = R$ (assuming $R > 0$) then $F(\rho, \phi, \theta) = \rho$ gives sphere as $F = R$. The normal is simply $\nabla F = e_\rho \frac{\partial \rho}{\partial \rho} + 0 + 0 = e_\rho$.

<u>Remark</u>: these examples are probably to simple to grasp the power of using sphericals or cylindricals. Trust me they're very useful, and much easier to understand geometrically for a problem that has spherical or cylindrical <u>symmetry</u>.

THE CURL OF A VECTOR FIELD:

We now find the next use of our friend "∇" (del or nabla). We are skipping ahead to §13.5 since it is our custom to differentiate before we integrate, and because it allows me to better explain certain topics in §13.2, 13.3, 13.4 and so on...

Def^n/ Let $F = \langle P, Q, R\rangle$ where each component function is differentiable, and $F = \langle F_1, F_2, F_3\rangle$ as well,

$$\begin{aligned}
\text{curl}(F) &\equiv \nabla \times F \\
&\equiv \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \\
&\equiv \langle \partial_2 F_3 - \partial_3 F_2,\ \partial_3 F_1 - \partial_1 F_3,\ \partial_1 F_2 - \partial_2 F_1 \rangle \\
&\equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) \\
&\equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}
\end{aligned}$$

E127 Let $F = \langle x, y, z\rangle$ find $\nabla \times F$.

$$\nabla \times F = \left\langle \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y),\ \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z),\ \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right\rangle = 0.$$

Hmm... notice $F = \nabla\left[\frac{1}{2}(x^2 + y^2 + z^2)\right]$. So $\underline{F \text{ is conservative}}$.

E128 Let $F = \langle -y, x, 0\rangle$ we saw this in E122 on (361)

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \hat{i}\left(\partial_y(0) - \partial_z(x)\right) - \hat{j}\left(\partial_x(0) - \partial_z(y)\right) + \hat{k}\left(\partial_x x + \partial_y y\right)$$

$$\Rightarrow \boxed{\nabla \times F = 2\hat{k}}$$

Using the $F = \langle P, Q, R\rangle$ notation we have $P = -y$ & $Q = x$ you might note $\frac{\partial P}{\partial y} = -1$ whereas $\frac{\partial Q}{\partial x} = 1$ thus $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. We'll learn a little later this mean $\nexists\, U$ such that $F = \nabla U$. That is F is <u>not</u> "conservative"

<u>Remark:</u> we can deduce a few facts about the "curl" operation. It measures if the vector field twists or curls about some point. In $\boxed{E127}$ we found $\nabla \times F = 0$ for a vector field which could be written $F = \rho e_\rho$ so it's purely "radial" it diverges straight away from the origin. On the other hand in $\boxed{E128}$ we found $\nabla \times F = 2\hat{k}$ for the vector field $F = \langle -y, x, 0\rangle$ which circles the origin in the xy-plane. We could say the curl measures the <u>circulation</u> of a vector field. Finally, we noticed that if $F = \nabla U$ then $\nabla \times F = \nabla \times (\nabla U) = 0$. Intuitively this is appealing since $A \times A = 0$, however we know that result for an ordinary vector, <u>not</u> for a <u>vector of operators</u> which is what $\nabla = \hat{i}\,\partial_x + \hat{j}\,\partial_y + \hat{k}\,\partial_z$ is. It is true though, we'll prove it.

Th$^{\underline{m}}$/ If g has continuous $2^{\underline{nd}}$ order partials then $\nabla \times \nabla g = 0$

<u>Proof:</u> follows from direct calculation + Clairaut's Th$^{\underline{m}}$,

$$\begin{aligned}
\nabla \times \nabla g &= \nabla \times \left[\langle \partial_x g, \partial_y g, \partial_z g\rangle\right] \\
&= \langle \partial_y(\partial_z g) - \partial_z(\partial_y g),\ \partial_z(\partial_x g) - \partial_x(\partial_z g),\ \partial_x(\partial_y g) - \partial_y(\partial_x g)\rangle \\
&= \left\langle \frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 g}{\partial z \partial y},\ \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 g}{\partial x \partial z},\ \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y \partial x} \right\rangle \\
&= \langle 0, 0, 0 \rangle \quad \left(\begin{array}{l}\text{applying } \partial_i \partial_j g = \partial_j \partial_i g \text{ for} \\ \text{continuous fnct. } g.\end{array}\right)
\end{aligned}$$

<u>Corollary:</u> If F is conservative then $\nabla \times F = 0$

this means we can verify that F fails to be conservative if $\nabla \times F \neq 0$. It is not however gauranteed that if $\nabla \times F = 0$ then F is conservative.

<u>Remark:</u> F is said to be irrotational iff $\nabla \times F = 0$

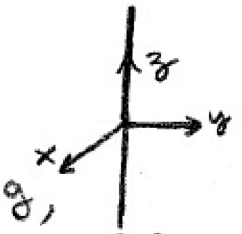
$$F \text{ conservative} \Longrightarrow \nabla \times F = 0 \ (\text{always})$$

$$\nabla \times F = 0 \Longrightarrow \underbrace{F = \nabla U}_{\text{it turns out the topology matters.}} \ (\text{sometimes})$$

Th^{m}/ If F is a vector field defined on <u>all</u> of $\mathbb{R}^3$ whose components functions have continuous partial derivatives and $\text{curl}(F) = 0$ then F is a conservative vector field

a stronger version says this Th^{m} holds for simply-connected domains of F. Simply-connected is a topological concept that means the space has no holes.

$\boxed{\text{E129}}$ Let $F = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$. Lets check if F is conservative. Later we'll show $\nabla \times F = 0$. Thus we should be able to find f such that $F = \nabla f$ meaning,



$x^2+y^2=0$
(z-axis)

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} = \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = \frac{\partial}{\partial x}\left[\tan^{-1}(y/x)\right]$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{\partial}{\partial y}\left[\tan^{-1}(y/x)\right]$$

Thus $f = \tan^{-1}(y/x)$. Here's the catch, $\text{dom}(F) = \{(x,y,z) \mid x^2+y^2 \neq 0\}$ whereas $\text{dom}(f) = \{(x,y,z) \mid x \neq 0\}$ thus F is <u>not</u> conservative despite the fact $\nabla \times F = 0$. This example is a subtle one and ultimately it leads to many deep advances in modern quantum field theory (its the magnetic monopole)

$\boxed{\text{E130}}$ Let $F = \langle 2x+y,\ z\cos(yz)+x,\ y\cos(yz)\rangle$. Is F conservative, if so find its potential function f such that $F = \nabla f$. We can test if F is conservative by examining $\nabla \times F$, the $\text{dom}(F) = \mathbb{R}^3$ so we'll not have to worry about the exception to the Th^{m}, in <u>this</u> <u>case</u>: $\nabla \times F = 0 \iff F$ conservative

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+y & z\cos(yz)+x & y\cos yz \end{vmatrix}$$

$$= \hat{i}\big(\cos(yz) - zy\sin(yz) - \cos(yz) + zy\sin(yz)\big) + \hat{j}(0-0) + \hat{k}(1-1)$$

$$= \underline{0 = \nabla \times F} \implies \text{since } \text{dom}(F) = \mathbb{R}^3$$

we find F conservative.

E130 **Continued:** We found that F is conservative with the help of the $\nabla \times F$ test. In principle this is not necessary, we alternatively could simply assume that $\exists f$ such that $F = \nabla f$. If F is not conservative we'll not be sucessful in finding the potential function.

$$\nabla f = \langle 2x + y,\ z\cos(yz) + x,\ y\cos(yz) \rangle$$

yields three partial differential eq^{n}'s, (PDEs)

$$\frac{\partial f}{\partial x} = 2x + y, \quad \frac{\partial f}{\partial y} = z\cos(yz) + x, \quad \frac{\partial f}{\partial z} = y\cos(yz)$$

generally PDEs aren't easy to solve, but these are easy. I'll illustrate a procedure you should <u>know</u> (this comes up in other math courses like Differential Eq^{n}'s and applications)

$$f = \int \frac{\partial f}{\partial x}\,\partial x = \int (2x + y)\,\partial x = x^2 + xy + C_1(y, z)$$

I've used the notation "∂x" to emphasize that we hold y & z constant in the integration, the constant with respect to x can be a function of y & z as I emphasize with the notation $C_1(y, z)$. Now we'll use the remaining two PDEs to pin down the explicit form of $C_1(y, z)$.

$$\frac{\partial f}{\partial y} = z\cos(yz) + x = \frac{\partial}{\partial y}\left[x^2 + xy + C_1(y, z)\right] = x + \frac{\partial C_1}{\partial y}$$

$$\Rightarrow z\cos(yz) = \frac{\partial C_1}{\partial y}$$

$$\Rightarrow C_1 = \int \frac{\partial C_1}{\partial y}\,\partial y = \int z\cos(yz)\,\partial y = \frac{z\sin(yz)}{z} + C_2(z).$$

Notice that $C_1 = C_1(y, z)$ so I knew that $C_2 = C_2(z)$, it can only depend on z otherwise C_1 might aquire an x-dependence.

$$\frac{\partial f}{\partial z} = y\cos(yz) = \frac{\partial}{\partial z}\left[x^2 + xy + \sin(yz) + C_2(z)\right]$$

$$\Rightarrow y\cos(yz) = y\cos(yz) + \frac{\partial C_2}{\partial z} \quad \Rightarrow \frac{dC_2}{dz} = 0 \quad \therefore \underline{C_2 = \text{constant}}$$

Here $\partial C_2/\partial z = dC_2/dz$ since C_2 only depends on z. In total,

$$\boxed{f = x^2 + xy + \sin(yz) + C_2 \quad \text{is the potential function for } F}$$

<u>Remark:</u> this seems longer than it is, I've put extra comments here to try to clarify the method. Also see §13.5 # 13 & 16 for more examples.

The Divergence of a Vector Field

Again we use the vector of operators ∇ to define,

Def^n/ Let F be a vector field with differentiable component fncts,

$$\text{div}(F) \equiv \nabla \cdot F \equiv \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

where $F = \langle P, Q, R \rangle = \langle F_1, F_2, F_3 \rangle$.

E131 Let $F = \langle x, y, z \rangle$ then

$$\nabla \cdot F = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

E132 Let $G = \langle -y, x, 0 \rangle$ then

$$\nabla \cdot G = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

__Langauge:__ the vector fields considered in **E127**, **E128**, **E131**, **E132** are examples of __irrotational__ and __incompressible__ vector fields. Speficially,

$$F = \langle x, y, z \rangle \text{ has } \nabla \cdot F = 3 \ \& \ \underline{\nabla \times F = 0}, \ \underline{F \text{ is irrotational}}$$

$$G = \langle -y, x, 0 \rangle \text{ has } \nabla \times G = 2\hat{k} \ \& \ \underline{\nabla \cdot G = 0}, \ \underline{G \text{ is incompressible}}$$

__Remark:__ the terminology "irrotational" and "incompressible" stem from early applications of vector fields to fluid flow. If V is the velocity field of a fluid then $\nabla \times V \neq 0 \Rightarrow$ a little paddle wheel will spin at the location where $\nabla \times V \neq 0$. If $\nabla \cdot V \neq 0$ that indicates the fluid is flowing in or out of the infinitesimal volume where $\nabla \cdot V$ is non zero. Electromagnetism shares many analogies to fluid flow, for example,

$$\underbrace{\nabla \cdot J}_{\substack{\text{flow of current} \\ \text{in/out of } dV}} = \underbrace{-\frac{\partial \rho}{\partial t}}_{\substack{\text{change in charge} \\ \text{density in } dV}}$$

this eq^n expresses the __conservation of charge__ locally.

Th^m/ If $F = \langle P, Q, R\rangle$ is a vector field such that P, Q, R have continuous 2^{nd} order partial derivatives then

$$\text{div}(\text{curl}(F)) = \nabla \cdot (\nabla \times F) = 0$$

Proof: follows from Clauriaut's Th^m again, use $F = \langle F_1, F_2, F_3\rangle$

$$\begin{aligned}\nabla \cdot (\nabla \times F) &= \nabla \cdot \langle \partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1\rangle \\ &= \cancel{\partial_1\partial_2 F_3} - \cancel{\partial_1\partial_3 F_2} + \cancel{\partial_2\partial_3 F_1} - \cancel{\partial_2\partial_1 F_3} + \cancel{\partial_3\partial_1 F_2} - \cancel{\partial_3\partial_2 F_1} \\ &= 0 //.\end{aligned}$$

$\boxed{E133}$ Is it possible that $F = \langle xz, xyz, -y^2\rangle = \nabla \times G$ for some G? Notice if this were true then $\nabla \cdot F = \nabla \cdot (\nabla \times G) = 0$ however $\nabla \cdot F \neq 0$ as we may easily calculate,

$$\nabla \cdot F = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz \neq 0.$$

$\boxed{E134}$ One of Maxwells Eq^n's can be written as $\nabla \cdot B = 0$, find another vector field which automatically solves this eq^n. The so-called vector potential A does the job, it is req^d that

$$B = \nabla \times A \qquad \text{thus} \qquad \nabla \cdot B = \nabla \cdot (\nabla \times A) = 0.$$

Notice that A is not unique since $\nabla \times \nabla g = 0$ we'll find the same magnetic field B from A or $A + \nabla g$.

Laplace Operator

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

this can be applied to real-valued functions or vectors.

$\boxed{E135}$ Gauss Law states $\nabla \cdot E = \rho/\epsilon_0$ and in electrostatics the electric field is given by the potential V according to $E = -\nabla V$ thus Gauss Law becomes $\nabla \cdot (-\nabla V) = \rho/\epsilon_0$ this gives $\nabla^2 V = -\rho/\epsilon_0$.

$\boxed{E136}$ In §13.5 #36 we encounter the Laplace Operator acting on vector fields

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \& \quad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

the sol^n's to these eq^n's are electromagnetic waves, that is light.

Curl and Divergence in Curved Coordinates

We found the gradients already, this will hurt a little. We begin with the divergence in cylindricals, (r, θ, z), we need a few preliminary facts

$$e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad x = r\cos\theta$$
$$e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad y = r\sin\theta$$
$$e_z = \hat{k} \qquad z = z$$

Then from the chain-rule we find

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

We need to solve these for $\partial/\partial x$ and $\partial/\partial y$.

$$\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} \qquad \text{note} \qquad \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}^{-1} = \frac{1}{r\cos^2\theta + r\sin^2\theta}\begin{pmatrix}r\cos\theta & -\sin\theta\\ r\sin\theta & \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} = \frac{1}{r}\begin{pmatrix}r\cos\theta & -\sin\theta\\ r\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix} \Rightarrow \boxed{\begin{aligned}\partial_x &= \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\\ \partial_y &= \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\end{aligned}}$$

Calculate,

$$\nabla\cdot F = \left(\hat{i}\,\partial_x + \hat{j}\,\partial_y + \hat{k}\,\partial_z\right)\cdot\left(F_r e_r + F_\theta e_\theta + F_z e_z\right)$$

$$= \left[\hat{i}\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right) + \hat{j}\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right) + \hat{k}\,\partial_z\right]\cdot$$
$$\cdot\left[\hat{i}\left(F_r\cos\theta - F_\theta\sin\theta\right) + \hat{j}\left(F_r\sin\theta - F_\theta\cos\theta\right) + \hat{k}F_z\right]$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(F_r\cos\theta - F_\theta\sin\theta\right) +$$
$$+\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(F_r\sin\theta + F_\theta\cos\theta\right) + \frac{\partial F_z}{\partial z}$$

$$= \underbrace{\cos\theta\frac{\partial}{\partial r}\left[F_r\cos\theta - F_\theta\sin\theta\right]}_{\mathrm{I}} - \underbrace{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left[F_r\cos\theta - F_\theta\sin\theta\right]}_{\mathrm{II}}$$

$$+ \underbrace{\sin\theta\frac{\partial}{\partial r}\left[F_r\sin\theta + F_\theta\cos\theta\right]}_{\mathrm{III}} + \underbrace{\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left[F_r\sin\theta + F_\theta\cos\theta\right]}_{\mathrm{IV}} + \frac{\partial F_z}{\partial z}$$

Continuing to find $\nabla \cdot F$ in cylindrical coordinates. We focus on the pieces I, II, III, IV one at a time. To begin we pull out constant pieces. Remember generally F_r and F_θ are functions of both r, θ, and z.

$$\text{I} = \cos^2\theta \frac{\partial F_r}{\partial r} - \cancel{\cos\theta\sin\theta \frac{\partial F_\theta}{\partial r}}^{\,(1)}$$

$$\text{II} = -\frac{\sin\theta}{r}\left[\frac{\partial F_r}{\partial \theta}\cos\theta - F_r\sin\theta - \frac{\partial F_\theta}{\partial \theta} - F_\theta\cos\theta\right]$$

$$= \cancel{-\frac{\sin\theta\cos\theta}{r}\frac{\partial F_r}{\partial \theta}}^{\,(3)} + \frac{\sin^2\theta}{r}F_r + \frac{\sin^2\theta}{r}\frac{\partial F_\theta}{\partial \theta} + \cancel{\frac{\sin\theta\cos\theta}{r}F_\theta}^{\,(2)}$$

$$\text{III} = \sin^2\theta \frac{\partial F_r}{\partial r} + \cancel{\sin\theta\cos\theta \frac{\partial F_\theta}{\partial r}}^{\,(1)}$$

$$\text{IV} = \frac{\cos\theta}{r}\left[\frac{\partial F_r}{\partial \theta}\sin\theta + F_r\cos\theta + \frac{\partial F_\theta}{\partial \theta} - F_\theta\sin\theta\right]$$

$$= \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial F_r}{\partial \theta}}^{\,(3)} + \frac{\cos^2\theta}{r}F_r + \frac{\cos^2\theta}{r}\frac{\partial F_\theta}{\partial \theta} - \cancel{\frac{\sin\theta\cos\theta}{r}F_\theta}^{\,(2)}$$

Now lets assemble $\nabla \cdot F$ given the above results, notice certain terms cancel,

$$\nabla \cdot F = (\cos^2\theta + \sin^2\theta)\frac{\partial F_r}{\partial r} + \frac{1}{r}(\sin^2\theta + \cos^2\theta)F_r + \frac{\sin^2\theta + \cos^2\theta}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \qquad (*)$$

$$= \frac{1}{r}\left[r\frac{\partial F_r}{\partial r} + F_r + \frac{\partial F_\theta}{\partial \theta} + r\frac{\partial F_z}{\partial z}\right]$$

$$= \boxed{\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial}{\partial z}(rF_z)\right] = \nabla \cdot F}$$

The last formula and $(*)$ are probably both useful. Notice that on $\mathbb{R}^2$ for $F = F_r e_r + F_\theta e_\theta = F_1\hat{i} + F_2\hat{j}$ we will find the same result without the z-terms,

$$\boxed{\nabla \cdot F = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}}$$

$$\boxed{\begin{array}{l} \text{Th}^{m}/ \quad \nabla\cdot F = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left[\rho^2 F_\rho\right] + \frac{1}{\rho\sin\varphi}\frac{\partial}{\partial\varphi}\left[\sin\varphi\, F_\varphi\right] + \frac{1}{\rho\sin\varphi}\frac{\partial F_\theta}{\partial\theta} \\ \text{where } F = F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta \text{ in spherical coordinates } \rho, \varphi, \theta \\ \text{where } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \varphi \leq \pi \end{array}}$$

Proof: We'll convert the def^{n} $\nabla\cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$ to sphericals. Lets recall what we already know from earlier work, (*) 365

$$F_x = \cos\theta\sin\varphi\, F_\rho + \cos\theta\cos\varphi\, F_\phi - \sin\theta\, F_\theta$$

$$F_y = \sin\theta\sin\varphi\, F_\rho + \sin\theta\cos\varphi\, F_\phi + \cos\theta\, F_\theta$$

$$F_z = \cos\varphi\, F_\rho - \sin\varphi\, F_\phi$$

We also calculated that

$$\nabla = e_\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho}e_\phi\frac{\partial}{\partial\phi} + \frac{e_\theta}{\rho\sin\phi}\frac{\partial}{\partial\theta} \quad : (*) \text{ on } 367$$

$$= \left(\hat{\imath}\cos\theta\sin\varphi + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\varphi\,\hat{k}\right)\frac{\partial}{\partial\rho} \quad : \text{using } (*) \text{ on } 364$$

$$\frac{1}{\rho}\left(\hat{\imath}\cos\theta\cos\varphi + \hat{\jmath}\sin\theta\cos\varphi - \sin\varphi\,\hat{k}\right)\frac{\partial}{\partial\phi}$$

$$\frac{1}{\rho\sin\phi}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)\frac{\partial}{\partial\theta}$$

$$\left.\begin{array}{ll} = \hat{\imath}\left[\cos\theta\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\theta\cos\varphi\frac{\partial}{\partial\phi} - \frac{\sin\theta}{\rho\sin\phi}\frac{\partial}{\partial\theta}\right] & \text{I} \\ + \hat{\jmath}\left[\sin\theta\sin\varphi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\sin\theta\cos\varphi\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\rho\sin\phi}\frac{\partial}{\partial\theta}\right] & \text{II} \\ + \hat{k}\left[\cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\phi}\right] & \text{III} \end{array}\right\} \star$$

$$= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

We could derive $\star$ from the chain rule, I took an alternative route. Now $F = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ and we may compute $\nabla\cdot F = \text{I}\, F_x + \text{II}\, F_y + \text{III}\, F_z$ where F_x, F_y, F_z are expressed in terms of ρ, φ, θ. We'll work out the gory details.

Proof Continued : $\nabla \cdot F$ is SPHERICAL COORDINATES

$$\text{I}\quad F_x = \cos\theta\sin\varphi\frac{\partial}{\partial\rho}\left[\cos\theta\sin\varphi\, F_\rho + \cos\theta\cos\phi\, F_\phi - \sin\theta\, F_\theta\right]$$
$$+\frac{1}{\rho}\cos\theta\cos\varphi\frac{\partial}{\partial\phi}\left[\cos\theta\sin\varphi\, F_\rho + \cos\theta\cos\phi\, F_\phi - \sin\theta\, F_\theta\right]$$
$$-\frac{\sin\theta}{\rho\sin\phi}\frac{\partial}{\partial\theta}\left[\cos\theta\sin\phi\, F_\rho + \cos\theta\cos\phi\, F_\phi - \sin\theta\, F_\theta\right]$$

$$= \cos^2\theta\sin^2\varphi\frac{\partial F_\rho}{\partial\rho} + \cos^2\theta\sin\phi\cos\phi\frac{\partial F_\phi}{\partial\rho} - \cos\theta\sin\theta\sin\phi\frac{\partial F_\theta}{\partial\rho}$$
$$+\frac{1}{\rho}\cos^2\theta\cos\phi\left[\cos\varphi F_\rho + \sin\varphi\frac{\partial F_\rho}{\partial\phi} - \sin\phi F_\phi + \cos\phi\frac{\partial F_\phi}{\partial\phi}\right] - \frac{\cos\theta\sin\theta\cos\phi}{\rho}\frac{\partial F_\theta}{\partial\phi} \quad ①$$
$$-\frac{\sin\theta}{\rho}\left[-\sin\theta F_\rho + \cos\theta\frac{\partial F_\rho}{\partial\theta}\right] - \frac{\sin\theta\cos\phi}{\rho\sin\phi}\left[-\sin\theta F_\phi + \cos\theta\frac{\partial F_\phi}{\partial\theta}\right]$$
$$-\frac{\sin\theta}{\rho\sin\phi}\left[-\cos\theta F_\theta - \sin\theta\frac{\partial F_\theta}{\partial\theta}\right]$$

$$\text{II}\quad F_y = \sin\theta\sin\varphi\frac{\partial}{\partial\rho}\left[\sin\theta\sin\phi\, F_\rho + \sin\theta\cos\phi\, F_\phi + \cos\theta\, F_\theta\right]$$
$$+\frac{\sin\theta\cos\varphi}{\rho}\frac{\partial}{\partial\phi}\left[\sin\theta\sin\phi\, F_\rho + \sin\theta\cos\phi\, F_\phi + \cos\theta\, F_\theta\right]$$
$$+\frac{\cos\theta}{\rho\sin\phi}\frac{\partial}{\partial\theta}\left[\sin\theta\sin\phi\, F_\rho + \sin\theta\cos\phi\, F_\phi + \cos\theta\, F_\theta\right]$$

$$= \sin^2\theta\sin^2\phi\frac{\partial F_\rho}{\partial\rho} + \sin^2\theta\sin\phi\cos\phi\frac{\partial F_\phi}{\partial\rho} + \sin\theta\cos\theta\sin\phi\frac{\partial F_\theta}{\partial\rho}$$
$$+\frac{\sin^2\theta\cos\phi}{\rho}\left[\cos\phi F_\rho + \sin\phi\frac{\partial F_\rho}{\partial\phi} - \sin\phi F_\phi + \cos\phi\frac{\partial F_\phi}{\partial\phi}\right] + \frac{\cos\theta\sin\theta\cos\phi}{\rho}\frac{\partial F_\theta}{\partial\phi} \quad ①$$
$$+\frac{\cos\theta}{\rho}\left[\cos\theta F_\rho + \sin\theta\frac{\partial F_\rho}{\partial\theta}\right] + \frac{\cos\theta\cos\phi}{\rho\sin\phi}\left[\cos\theta F_\phi + \sin\theta\frac{\partial F_\phi}{\partial\theta}\right]$$
$$+\frac{\cos\theta}{\rho\sin\phi}\left[-\sin\theta F_\theta + \cos\theta\frac{\partial F_\theta}{\partial\theta}\right]$$

$$\text{III}\quad F_z = \cos\varphi\frac{\partial}{\partial\rho}\left[\cos\varphi\cdot F_\rho - \sin\varphi F_\phi\right] - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\left[\cos\phi F_\rho - \sin\phi F_\phi\right]$$
$$= \cos^2\varphi\frac{\partial F_\rho}{\partial\rho} - \sin\varphi\cos\varphi\frac{\partial F_\phi}{\partial\rho} + \frac{\sin^2\phi}{\rho}F_\rho - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial F_\rho}{\partial\phi} + \cdots$$
$$+\frac{\sin\phi\cos\phi}{\rho}F_\phi + \frac{\sin^2\phi}{\rho}\frac{\partial F_\phi}{\partial\phi}$$

Proof Continued: sum ① F_x + ② F_y + ③ F_z using appropriate trig-identity and cancelations,

$$\begin{aligned}
\nabla \cdot F = \; & \frac{\partial F_\rho}{\partial \rho}\left[\cos^2\theta \sin^2\varphi + \sin^2\theta \sin^2\phi + \cos^2\phi\right] \\
& + \frac{\partial F_\phi}{\partial \rho}\left[\cos^2\theta \sin\phi\cos\phi + \sin^2\theta \sin\phi\cos\phi - \sin\phi\cos\phi\right] \\
& + \frac{\partial F_\theta}{\partial \rho}\left[-\cos\theta\sin\theta\sin\phi + \sin\theta\cos\theta\sin\phi\right] \\
& + \frac{\partial F_\rho}{\partial \phi}\left[\frac{1}{\rho}\cos^2\theta\cos\phi\sin\phi + \frac{1}{\rho}\sin^2\theta\cos\phi\sin\phi - \frac{1}{\rho}\sin\phi\cos\phi\right] \\
& + \frac{\partial F_\phi}{\partial \phi}\left[\frac{1}{\rho}\cos^2\theta\cos^2\phi + \frac{1}{\rho}\sin^2\theta\cos^2\phi + \frac{\sin^2\phi}{\rho}\right] \\
& + \frac{\partial F_\rho}{\partial \theta}\left[-\frac{\sin\theta\cos\theta}{\rho} + \frac{\sin\theta\cos\theta}{\rho}\right] \\
& + \frac{\partial F_\phi}{\partial \theta}\left[-\frac{\sin\theta\cos\theta}{\rho} + \frac{\cos\theta\sin\theta}{\rho}\right] \\
& + \frac{\partial F_\theta}{\partial \theta}\left[+\frac{\sin^2\theta}{\rho\sin\phi} + \frac{\cos^2\theta}{\rho\sin\phi}\right] \\
& + F_\rho \frac{1}{\rho}\left[\cos^2\theta\cos^2\phi + \sin^2\theta + \sin^2\theta\cos^2\phi + \cos^2\theta + \sin^2\phi\right] \\
& + F_\phi \frac{1}{\rho}\left[-\cancel{\cos^2\theta\cos\phi\sin\phi} + \frac{\sin^2\theta\cos\phi}{\sin\phi} - \cancel{\sin^2\theta\cos\phi\sin\phi} + \frac{\cos^2\theta\cos\phi}{\sin\phi} + \cancel{\sin\phi\cos\phi}\right] \\
& + F_\theta \frac{1}{\rho}\left[\cancel{\frac{\sin\theta\cos\theta}{\sin\phi}} - \cancel{\frac{\sin\theta\cos\theta}{\sin\phi}}\right]
\end{aligned}$$

$$= \left(\frac{\partial F_\rho}{\partial \rho} + \frac{2}{\rho}F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\cos\phi}{\rho\sin\phi}F_\phi + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial \theta}$$

Then notice that

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 F_\rho\right] = \frac{2\rho}{\rho^2}F_\rho + \frac{\rho^2}{\rho^2}\frac{\partial F_\rho}{\partial \rho} = \frac{2}{\rho}F_\rho + \frac{\partial F_\rho}{\partial \rho}$$

$$\frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}\left[\sin\phi\, F_\phi\right] = \frac{\cos\phi}{\rho\sin\phi}F_\phi + \frac{1}{\rho}\frac{\sin\phi}{\sin\phi}\frac{\partial F_\phi}{\partial \phi} = \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\cos\phi}{\rho\sin\phi}F_\phi$$

$$\therefore \boxed{\nabla \cdot F = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 F_\rho\right] + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}\left[\sin\phi\, F_\phi\right] + \frac{1}{\rho\sin\phi}\frac{\partial F_\theta}{\partial \theta}}$$

$$\boxed{\begin{array}{l} \text{Th}^{\text{m}}/ \quad \nabla\times F = \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)e_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)e_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}[rF_\theta] - \frac{\partial F_r}{\partial\theta}\right)e_z \\ \text{where } F = F_r e_r + F_\theta e_\theta + F_z e_z \text{ in cylindrical coordinates.} \end{array}}$$

<u>Proof:</u> the definition was given in CARTESIAN'S, focus on $e_r \cdot (\nabla\times F)$

$$\begin{aligned}
(\nabla\times F)\cdot e_r &= (\nabla\times F)\cdot(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \\
&= \cos\theta\,(\nabla\times F)\cdot\hat{i} + \sin\theta\,(\nabla\times F)\cdot\hat{j} \\
&= \cos\theta\,[\partial_y F_z - \partial_z F_y] + \sin\theta\,[\partial_z F_x - \partial_x F_z] \\
&= \underbrace{\cos\theta\,\partial_y F_z}_{\mathrm{I}}\ \underbrace{-\cos\theta\,\partial_z F_y}_{\mathrm{II}}\ \underbrace{+\sin\theta\,\partial_z F_x}_{\mathrm{III}}\ \underbrace{-\sin\theta\,\partial_x F_z}_{\mathrm{IV}}
\end{aligned}$$

We'll proceed to convert I, II, III, IV to cylindricals, need to convert the partial derivatives, we already have formulas for F_x, F_y, F_z in terms of F_r, F_θ, F_z (see (*) on 363) and likewise for $\partial_x, \partial_y, \partial_z$ in terms of $\partial_r, \partial_\theta, \partial_z$ on 375. Calculate then,

$$\mathrm{I} = \cos\theta\left[\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right][F_z] = \overset{①}{\cancel{\cos\theta\sin\theta\,\partial_r F_z}} + \frac{\cos^2\theta}{r}\partial_\theta F_z$$

$$\mathrm{II} = -\cos\theta\,[\partial_z][F_r\sin\theta + F_\theta\cos\theta] = \overset{③}{\cancel{-\cos\theta\sin\theta\,\partial_z F_r}} - \cos^2\theta\,\partial_z F_\theta$$

$$\mathrm{III} = \sin\theta\,[\partial_z][F_r\cos\theta - F_\theta\sin\theta] = \overset{②}{\cancel{\cos\theta\sin\theta\,\partial_z F_r}} - \sin^2\theta\,\partial_z F_\theta$$

$$\mathrm{IV} = -\sin\theta\left[\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right][F_z] = \overset{①}{\cancel{-\sin\theta\cos\theta\,\partial_r F_z}} + \frac{\sin^2\theta}{r}\partial_\theta F_z$$

We see some encouraging cancellations, we find,

$$\boxed{(\nabla\times F)\cdot e_r = \frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}}$$

One component down, two to go.

- Our strategy here is to work on the curl one component at a time, this makes the calculation more manageable.

Proof Continued for $\nabla \times F$ in polar coordinates, the θ-component

$$(\nabla\times F)\cdot e_\theta = (\nabla\times F)\cdot[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}]$$
$$= -\sin\theta\,(\nabla\times F)\cdot\hat{i} + \cos\theta\,(\nabla\times F)\cdot\hat{j}$$
$$= \underbrace{-\sin\theta\,\partial_y F_z}_{\mathrm{I}} + \underbrace{\sin\theta\,\partial_z F_y}_{\mathrm{II}} + \underbrace{\cos\theta\,\partial_z F_x}_{\mathrm{III}} - \underbrace{\cos\theta\,\partial_x F_z}_{\mathrm{IV}}$$

Proceed as before, convert to cylindrical coordinates,

$$\mathrm{I} = -\sin\theta\left[\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right][F_z] = -\sin^2\theta\,\partial_r F_z - \frac{\sin\theta\cos\theta}{r}\partial_\theta F_z$$

$$\mathrm{II} = \sin\theta\,[\partial_z][F_r\sin\theta + F_\theta\cos\theta] = \sin^2\theta\,\partial_z F_r + \sin\theta\cos\theta\,\partial_z F_\theta$$

$$\mathrm{III} = \cos\theta\,[\partial_z][F_r\cos\theta - F_\theta\sin\theta] = \cos^2\theta\,\partial_z F_r - \sin\theta\cos\theta\,\partial_z F_\theta$$

$$\mathrm{IV} = -\cos\theta\left[\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right][F_z] = -\cos^2\theta\,\partial_r F_z + \frac{\sin\theta\cos\theta}{r}\partial_\theta F_z$$

After a few cancellations in $\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$,

$$\boxed{(\nabla\times F)\cdot e_\theta = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}}$$

And now the last component,

$$(\nabla\times F)\cdot e_z = \partial_x F_y - \partial_y F_x$$
$$= \left[\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\right][F_r\sin\theta + F_\theta\cos\theta]$$
$$- \left[\sin\theta\,\partial_r + \frac{1}{r}\cos\theta\,\partial_\theta\right][F_r\cos\theta - F_\theta\sin\theta]$$

$$= \Big[\cancel{\sin\theta\cos\theta\,\partial_r F_r}^{(1)} + \cos^2\theta\,\partial_r F_\theta$$
$$- \frac{1}{r}\sin^2\theta\,\partial_\theta F_r - \cancel{\frac{1}{r}\sin\theta\cos\theta\,F_r}^{(2)} - \cancel{\frac{1}{r}\sin\theta\cos\theta\,\partial_\theta F_\theta}^{(3)} + \frac{1}{r}\sin^2\theta\,F_\theta\Big]$$
$$- \Big[\cancel{\sin\theta\cos\theta\,\partial_r F_r}^{(1)} - \sin^2\theta\,\partial_r F_\theta + \frac{1}{r}\cos^2\theta\,\partial_\theta F_r - \cancel{\frac{1}{r}\cos\theta\sin\theta\,F_r}^{(2)}$$
$$+ \cancel{\frac{1}{r}\cos\theta\sin\theta\,\partial_\theta F_\theta}^{(3)} - \frac{\cos^2\theta}{r}F_\theta\Big]$$

$$= \partial_r F_\theta - \frac{1}{r}\partial_\theta F_r + \frac{1}{r}F_\theta = \boxed{\frac{1}{r}\frac{\partial}{\partial r}[rF_\theta] - \frac{1}{r}\frac{\partial F_r}{\partial\theta} = (\nabla\times F)\cdot e_z}$$

So we find that

$$\nabla\times F = [(\nabla\times F)\cdot e_r]e_r + [(\nabla\times F)\cdot e_\theta]e_\theta + [(\nabla\times F)\cdot e_z]e_z$$

$$\boxed{\nabla\times F = \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)e_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)e_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}[rF_\theta] - \frac{\partial F_r}{\partial\theta}\right)e_z}$$

BONUS POINT: Derive the formula for $\nabla\times F$ in Spherical coordinates, its on next page

SUMMARY OF DIFFERENTIAL VECTOR CALCULUS

This table is borrowed from Thomas' Calculus 10th ed. Notice that e_r, e_θ, e_ϕ are denoted u_r, u_θ, u_ϕ and e_ρ is u_ρ. We have derived much of what is here. You should be equipt to derive the formulas for the Laplacian $\nabla^2 f$ given the calculations I've shown you in these notes.

Vector Operator Formulas in Cartesian, Cylindrical, and Spherical Coordinates; Vector Identities

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. F_x, F_y, and F_z are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.	**Cylindrical** (r, θ, z) $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$ are unit vectors in the directions of increasing r, θ, and z. F_r, F_θ, and F_z are the scalar components of $\mathbf{F}(r, \theta, z)$ in these directions.	**Spherical** (ρ, ϕ, θ) $\mathbf{u}_\rho$, $\mathbf{u}_\phi$, and $\mathbf{u}_\theta$ are unit vectors in the directions of increasing ρ, ϕ, and θ. F_ρ, F_ϕ, and F_θ are the scalar components of $\mathbf{F}(\rho, \phi, \theta)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{u}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{u}_\phi + \frac{1}{\rho \sin\phi}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta$
Divergence	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F_\rho) + \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \phi}(F_\phi \sin\phi) + \frac{1}{\rho \sin\phi}\frac{\partial F_\theta}{\partial \theta}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{1}{r}\mathbf{u}_r & \mathbf{u}_\theta & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & F_\theta & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{\mathbf{u}_\rho}{\rho^2 \sin\phi} & \frac{\mathbf{u}_\phi}{\rho \sin\phi} & \frac{\mathbf{u}_\theta}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho \sin\phi\, F_\theta \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$

Vector Triple Products

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Vector Identities for the Cartesian Form of the Operator ∇

In the identities listed here, $f(x, y, z)$ and $g(x, y, z)$ are differentiable scalar functions and $\mathbf{u}(x, y, z)$ and $\mathbf{v}(x, y, z)$ are differentiable vector functions.

$$\nabla \cdot f\mathbf{v} = f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f = f\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla) f$$

$$\nabla \times f\mathbf{v} - f\nabla \times \mathbf{v} + \nabla f \times \mathbf{v}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u})$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v})$$

PHYSICS' CONVENTIONS FOR VECTOR CALCULUS

Probably some of you will encounter the other type of spherical & cylindrical coordinates. In my experience physicists will always use these conventions where we have the following dictionary

Math	Physics
ρ	$r = \sqrt{x^2+y^2+z^2}$
θ	ϕ , $0 \leq \phi \leq 2\pi$
ϕ	θ , $0 \leq \theta \leq \pi$
r	$s = \sqrt{x^2+y^2}$

For your convenience I include a few results borrowed from David Griffith's EXCELLENT (!) text on Electromagnetics.

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

PHYSICS' CONVENTIONS CONTINUED

There is nothing really new here, we just take what we did and change notation according to the dictionary. These formulas are taken from the cover of Griffith's E&M text.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$