Stewart discusses these in §10.5 and §11.4 p. 777, §12.6 and §13.6. I introduced the idea of a parametric surface & its tangent plane on (317-319). The surface integral of a vector field is phrased in terms of a parametric description of the surface, it is thus important for us to review what a parametric surface is.

> **Def^{n}/** A parametrized surface S consists of a parameter space $D \subseteq \mathbb{R}^2$ and a mapping Σ which is mostly invertible $\Sigma : D \longrightarrow S \subset \mathbb{R}^3$, $(u,v) \longmapsto \Sigma(u,v)$. Where
> $$\Sigma(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$
> we say S is oriented if the <u>normal</u> <u>vector</u> field $N(u,v)$ is nonzero $\forall (u,v) \in D$, where
> $$N(u,v) \equiv \frac{\partial \Sigma}{\partial u} \times \frac{\partial \Sigma}{\partial v}$$
> we may also use $r(u,v)$ as the mapping in which case the normal vector field would be $N(u,v) = r_u \times r_v$.

$\boxed{E151}$ $\Sigma(\theta,\varphi) = R(\cos\theta \sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi)$ with $R > 0$ and $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$ parametrizes a sphere. (see $\boxed{E78}$ (317))

$$\frac{\partial \Sigma}{\partial \theta} = R\langle -\sin\theta\sin\varphi,\ \cos\theta\sin\varphi,\ 0\rangle$$

$$\frac{\partial \Sigma}{\partial \varphi} = R\langle \cos\theta\cos\varphi,\ \sin\theta\cos\varphi,\ -\sin\varphi\rangle$$

$$N(\theta,\varphi) = R^2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix} = R^2\langle -\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi,$$

$$= R^2\langle -\cos\theta\sin^2\varphi,\ -\sin\theta\sin^2\varphi,\ -\sin^2\theta\sin\varphi\cos\varphi - \cos^2\theta\sin\varphi\cos\varphi\rangle$$

$$= R^2\sin\varphi\langle -\cos\theta\sin\varphi,\ -\sin\theta\sin\varphi,\ -\cos\varphi\rangle$$

$$= -(R^2\sin\varphi)\, e_\rho$$

see (*) on (364) to get $\left(e_\rho = \nabla\rho / |\nabla\rho|\right)$

$\therefore$ $\boxed{N(\theta,\varphi) = -(R^2\sin\varphi)\, e_\rho}$

this is the normal vector field to the sphere, since $\sin\varphi \ge 0$ for $0 \le \varphi \le \pi$ we note that $N(\theta,\varphi)$ points towards origin.

Def^{n}/ A closed surface S is the boundary of some solid region E ; $S = \partial E$. If the normal vector field points outward everywhere on S then we say S is positively oriented, if it points inward everywhere we say S has a negative orientation.

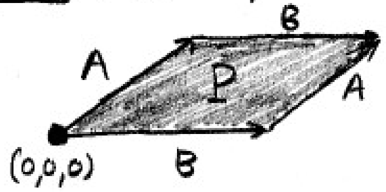
<u>Remark</u>: If $\mathbf{X}(u,v)$ is negatively oriented then $\mathbf{X}(v,u)$ is positively oriented. Switching the order of the parameters will switch $N(u,v) = \mathbf{X}_u \times \mathbf{X}_v$ to $N(v,u) = \mathbf{X}_v \times \mathbf{X}_u$.

E152 We found $\mathbf{X}(\theta,\varphi) = R(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ yields a negative orientation to the sphere since $N(\theta,\varphi) = -R^2\sin\varphi\, e_\rho$ points inward. To get a positive orientation for the sphere we need only switch the order of θ and φ. Then

$$N(\varphi,\theta) = \mathbf{X}_\varphi \times \mathbf{X}_\theta = -\mathbf{X}_\theta \times \mathbf{X}_\varphi = \boxed{R^2\sin\varphi\, e_\rho = N(\varphi,\theta)}$$

<u>Remark</u>: For many surfaces we cannot intrinsically choose a normal vector, I mean there are two choices. The plane is a good example, both $\langle a,b,c\rangle$ and $\langle -a,-b,-c\rangle$ provide a normal vector field, which should we choose? The answer is simple, we cannot choose, sometimes to make the problem unambiguous we must be told both the surface and the direction of its orientation. However, if we are given $\mathbf{X}(u,v)$ then $N(u,v)$ is implicitly given.

E153 Find a parametrization for the parallelogram pictured below.



$$\mathbf{X}(s,t) = sA + tB \quad \text{for} \quad 0 \le s,t \le 1$$

will shade out the region, just think about the vector addition. The normal will be $A \times B$

$$N(s,t) = \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} = A \times B.$$

this choice of parametrization gives the parallelogram P a normal pointing into the page.

Def^n/ Suppose $\Sigma: D \to S$ where $(u,v) \mapsto \Sigma(u,v)$, f continuous on S,

$$\iint_S f(x,y,z)\,dS = \iint_D f(\Sigma(u,v))\,|\Sigma_u \times \Sigma_v|\,dA$$

- we may refer to this as a scalar surface integral. It takes a weighted sum of the scalar function f over the surface.

Def^n/ The scalar surface integral of $f = 1$ over S gives the <u>surface area</u>

E154 Consider the sphere. Lets find its surface area. Notice

$$|N(\varphi,\theta)| = |\Sigma_\varphi \times \Sigma_\theta| = |R^2 \sin\varphi\, e_\rho| = R^2 \sin\varphi \quad \text{since } |e_\rho| = 1.$$

$$A(S^2) = \iint_{S^2} 1\,dS = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} R^2 \left(-\cos\varphi\Big|_0^{\pi}\right) d\theta, \quad -\cos\varphi\Big|_0^{\pi} = -\cos\pi + \cos(0)$$

$$= \int_0^{2\pi} 2R^2\, d\theta$$

$$= \boxed{4\pi R^2}$$

E155 If $f(x,y,z) = \sigma(x,y,z) =$ a mass density then the scalar surface integral calculates the total mass of the surface S. Suppose that $\sigma(x,y,z) = z^2$. Again consider the sphere S,

$$\iint_S \sigma(x,y,z)\,dS = \int_0^{2\pi}\int_0^{\pi} z^2 R^2 \sin\varphi\, d\varphi\, d\theta \qquad \underline{\text{but } z = R\cos\varphi}$$

$$= \int_0^{2\pi}\int_0^{\pi} R^4 \cos^2\varphi \sin\varphi\, d\theta$$

$$= (2\pi R^4)\left(-\tfrac{1}{3}\cos^3\varphi\Big|_0^{\pi}\right.$$

$$= (2\pi R^4)\left(-\tfrac{1}{3}\cos^3\pi + \tfrac{1}{3}\cos^3 0\right)$$

$$= \boxed{\frac{4\pi R^4}{3}} = \text{mass of the surface } S.$$

E156 The graph $z = f(x,y)$ can be cast as a parametric surface with parameters x & y. Let $(x,y) \in \text{dom}(f)$ then

$$\Phi(x,y) = (x, y, f(x,y))$$

$$\frac{\partial \Phi}{\partial x} = \langle 1, 0, \partial f/\partial x \rangle$$

$$\frac{\partial \Phi}{\partial y} = \langle 0, 1, \partial f/\partial y \rangle$$

$$N(x,y) = \left(\hat{i} + \frac{\partial f}{\partial x}\hat{k}\right) \times \left(\hat{j} + \frac{\partial f}{\partial y}\hat{k}\right) = \frac{\partial f}{\partial y}\hat{i}\times\hat{k} + \frac{\partial f}{\partial x}\hat{k}\times\hat{j} + \hat{i}\times\hat{j}$$

$$\therefore\ N(x,y) = \langle -\partial f/\partial x, -\partial f/\partial y, 1 \rangle$$

$$\Rightarrow \boxed{\text{Area of graph } f = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\ dA}$$

Consider $z = x^2 + y^2$ with $D = \{(x,y) \mid x^2 + y^2 \leq 9\}$ find the surface area of this paraboloid.

$$\text{Area} = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\ dA$$

: use polar coordinates to integrate, $x = r\cos\theta$, $y = r\sin\theta$
$x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$

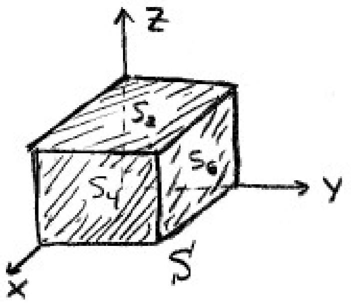
$$= \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\ r\,dr\,d\theta$$

$$= (2\pi)\frac{2}{3}\frac{1}{8}(4r^2+1)^{3/2}\Big|_0^3$$

$$= \boxed{\frac{\pi}{6}\left[(37)^{3/2} - 1\right]}$$

E157 Some surfaces are not smooth everywhere, but as long as they're piecewise smooth we can integrate. The unit cube for example. $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

$$\Sigma_1(y,x) = (x,y,0) \Rightarrow N_1 = \langle 0,0,-1\rangle$$
$$\Sigma_2(x,y) = (x,y,1) \Rightarrow N_2 = \langle 0,0,1\rangle$$
$$\Sigma_3(z,y) = (0,y,z) \Rightarrow N_3 = \langle -1,0,0\rangle$$
$$\Sigma_4(y,z) = (1,y,z) \Rightarrow N_4 = \langle 1,0,0\rangle$$
$$\Sigma_5(x,z) = (x,0,z) \Rightarrow N_5 = \langle 0,-1,0\rangle$$
$$\Sigma_6(z,x) = (x,1,z) \Rightarrow N_6 = \langle 0,1,0\rangle$$



I've given S a positive orientation. Calculate $\iint_S xyz\,dS$

$$\iint_S x\,dS = \iint_{S_1} xyz\,dS + \iint_{S_2} xyz\,dS + \iint_{S_3} xyz\,dS + \iint_{S_4} xyz\,dS + \iint_{S_5} xyz\,dS + \iint_{S_6} xyz\,dS$$

$$= \iint_{S_2} xyz\,dS + \iint_{S_4} xyz\,dS + \iint_{S_6} xyz\,dS$$

$$= \int_0^1\int_0^1 xy(1)\,dx\,dy + \int_0^1\int_0^1 (1)yz\,dy\,dz + \int_0^1\int_0^1 x(1)z\,dx\,dz$$

$$= \left(\frac{1}{2}x^2\Big|_0^1\right)\left(\frac{1}{2}y^2\Big|_0^1\right) + \frac{1}{4} + \frac{1}{4}$$

$$= \boxed{3/4}$$

<u>Remark</u>: Example 3 on p. 951 is worth looking over.

We define $n = \frac{N}{|N|}$ to be the unit normal to the parametric surface. We know that $F \bullet n$ will measure the component of F which points along the normal to the surface,

Def^{n}/ If F is a continuous vector field defined on an oriented surface with unit normal n, then the <u>surface integral</u> or <u>flux</u> of F over S is

$$\iint_S F \bullet dS = \iint_S F \bullet n \, dS$$

we may also employ the notation $\iint_S \vec{F} \bullet d\vec{A}$. When S is piecewise smooth we take the sum of the pieces.

Suppose we have the parametrization $\Sigma(u,v)$ then

$$\begin{aligned}
\iint_S F \bullet dS &= \iint_S F \bullet n \, dS \\
&= \iint_D F(\Sigma(u,v)) \bullet \left(\frac{\Sigma_u \times \Sigma_v}{|\Sigma_u \times \Sigma_v|} \right) |\Sigma_u \times \Sigma_v| \, du \, dv \\
&= \boxed{\iint_D F(\Sigma(u,v)) \bullet (\Sigma_u \times \Sigma_v) \, du\,dv = \iint_S F \bullet dS} \quad (*)
\end{aligned}$$

E158 Suppose $\vec{J} = \frac{\text{current}}{\text{area}}$ then $\vec{J} \bullet d\vec{A}$ will give us the current that cuts through the area perpendicularly. In particular Suppose $\vec{J} = r\hat{k}$ for $0 \leq r \leq a$ and $\vec{J} = 0$ for $r > a$ find the current through the xy-plane. (call it S)

$$\begin{aligned}
I = \iint_S \vec{J} \bullet d\vec{A} &= \int_0^{2\pi}\int_0^a (r\hat{k}) \bullet (r\,dr\,d\theta\,\hat{k}) \\
&= \int_0^{2\pi} d\theta \int_0^a r^2 \, dr \\
&= \boxed{2\pi a^3/3 = I_{\text{enclosed by } xy\text{-plane}} = I_{enc.}}
\end{aligned}$$

: choosing the upward orientation for xy-plane $d\vec{A} = (r\,dr\,d\theta)\hat{k}$

<u>Challenge</u>: find I_{enc} by $ax + by + cz = 0$, same current density.

Remark: in $\boxed{E158}$ I pretty much ignored the def^{n} and applied what I would call the geometrical approach. It does match up with the sol^{n} written straight from the def^{n} lets see how,

$\boxed{E159}$ Again suppose $\vec{J} = \langle 0, 0, r \rangle$ for $r \leq a$ and $\vec{J} = 0$ for $r > a$, we're using cylindrical coordinates.

$$\Phi(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 0\rangle \quad , \quad 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi$$

this parametrizes the xy-plane where $\vec{J} \neq 0$, call it $\tilde{S}$

$$\begin{aligned}
\Phi_r \times \Phi_\theta &= \langle \cos\theta, \sin\theta, 0\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= \langle 0, 0, r\cos^2\theta + r\sin^2\theta \rangle \\
&= \langle 0, 0, r \rangle = r\hat{k}
\end{aligned}$$

Then apply the definition to calculate $\iint_{\tilde{S}} \vec{J} \cdot dA$ ($\tilde{S}$ is the disk of radius a in xy-plane.)

$$\begin{aligned}
I &= \iint_{\tilde{S}} \langle 0,0,r\rangle \cdot dS \\
&= \int_0^{2\pi}\int_0^a \langle 0,0,r\rangle \cdot \langle 0,0,r\rangle\, dr\, d\theta \quad : \text{using } (*) \text{ on } 407 \\
&= \int_0^{2\pi}\int_0^a r^2\, dr\, d\theta \\
&= 2\pi a^3/3.
\end{aligned}$$

Comment: The infinitesimal vector area element of the surface is what I called $d\vec{A}$. We observe

$$d\vec{A} = (\Phi_u \times \Phi_v)\, du\, dv$$

Then you can see my "geometrical approach" is just another way of assembling the surface integral.

$$\iint_S F \cdot dA = \iint_D F \cdot (\Phi_u \times \Phi_v)\, du\, dV = \iint_S F \cdot dS$$

$\boxed{E160}$ Let $\vec{E} = \dfrac{Q}{4\pi\epsilon_0}\dfrac{e_\rho}{\rho^2}$ where Q, ϵ_0 are constants.
Find the flux of E through a sphere of radius R called S.
From $\boxed{E152}$ we recall $\Sigma(\varphi,\theta) = (R\cos\theta\sin\varphi,\ R\sin\theta\sin\varphi,\ R\cos\varphi)$ parametrizes the sphere where $0 \leq \varphi \leq \pi$ & $0 \leq \theta \leq 2\pi$ and we calculated that $\Sigma_\varphi \times \Sigma_\theta = R^2\sin\varphi\, e_\rho$ we find that the <u>vector area element to the sphere</u> is $\boxed{d\vec{A} = (R^2\sin\varphi\, d\varphi\, d\theta)\, e_\rho}$

$$\Phi_E \equiv \iint_S \vec{E}\cdot d\vec{A} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{Q}{4\pi\epsilon_0 R^2}e_\rho\right)\cdot\left(R^2\sin\varphi\, d\varphi\, d\theta\, e_\rho\right) \quad : \text{ noting } \rho = R \text{ on } S.$$

$$= \frac{Q}{4\pi\epsilon_0}\underbrace{\int_0^{2\pi}d\theta}_{2\pi}\underbrace{\int_0^{\pi}\sin\varphi\, d\varphi}_{2} \qquad : e_\rho\cdot e_\rho = 1.$$

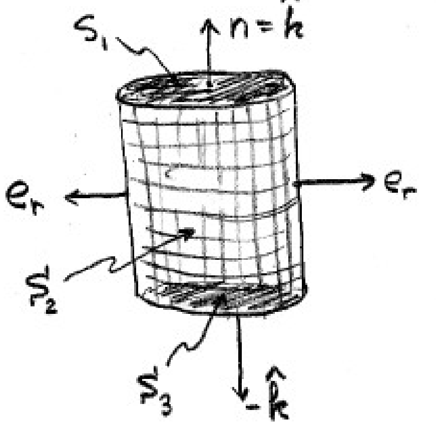
$$= \frac{Q}{4\pi\epsilon_0}\cdot 4\pi$$

$$= \boxed{Q/\epsilon_0 = \Phi_E}$$

this is an example of the integral form of Gauss' Law.

<u>Remark</u>: Spheres come up often. Its worth noting that $d\vec{A} = (R^2\sin\varphi\, d\varphi\, d\theta)\, e_\rho$. Also you should be aware of the concept of "solid angle", its denoted Ω and $d\Omega = \sin\varphi\, d\varphi\, d\theta$. The Ω_{total} for a sphere is 4π.

$\boxed{E161}$ Let $B = \dfrac{\mu_0 I}{2\pi r}e_\theta$ find the flux of B through a cylinder of radius a and height h with base on xy-plane



$$\left.\begin{aligned} d\vec{A}_1 &= (r\,dr\,d\theta)\,\hat{k} \\ d\vec{A}_2 &= (r\,d\theta\,dz)\,e_r \\ d\vec{A}_3 &= (r\,dr\,d\theta)(-\hat{k}) \end{aligned}\right\} \quad \text{all orthogonal to } e_\theta \quad \therefore\ \vec{B}\cdot d\vec{A} = 0$$

$$\boxed{\Phi_B = \iint_S \vec{B}\cdot d\vec{A} = 0}$$

this is an example of the integral form of the no magnetic monopoles eq$^{\underline{n}}$ $\nabla\cdot B = 0$. we'll see why soon.

E162 Let $F = \langle P, Q, R \rangle$ and suppose for $(x,y) \in D$ the surface S is a graph $z = g(x,y)$. In parametrized form

$$\Sigma(x,y) = (x, y, g(x,y))$$

$$\Sigma_x \times \Sigma_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1 \rangle$$

$$\iint_S F \cdot dS = \iint_D \langle P, Q, R \rangle \cdot \langle -g_x, -g_y, 1 \rangle \, dA$$

Consider then the case $F = \langle 0, y, -z \rangle$ and the surface S consists of the paraboliod $z = x^2+y^2$ and the disk at $z = 1$ which caps the paraboloid. Orient S inwards, call the disk S_2 and the paraboloid S_1 so $S = S_1 \cup S_2$.

$$\iint_{S_1} F \cdot dS = \iint_D \langle 0, y, -z \rangle \cdot \langle -2x, -2y, 1 \rangle \, dA$$

$$= \iint_D (-2y^2 - z) \, dA \qquad D = \{ (x,y) \mid x^2+y^2 \leq 1 \}$$
$$z = x^2+y^2 = r^2$$

$$= \int_0^{2\pi} \int_0^1 (-2r^2 \sin^2\theta - r^2) \, r \, dr \, d\theta$$

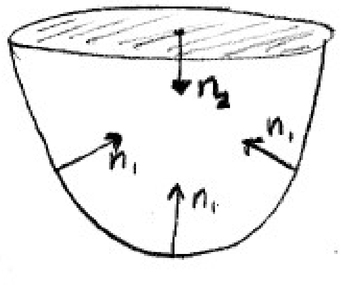
$$= \int_0^{2\pi} \left[-\tfrac{1}{2} r^4 \sin^2\theta - \tfrac{1}{4} r^4 \right|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{-1}{2} \sin^2\theta - \tfrac{1}{4} \right) d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{-1}{4}(1 - \cos(2\theta)) - \tfrac{1}{4} \right) d\theta$$

$$= \left(\tfrac{-1}{2}\theta + \tfrac{1}{8}\sin(2\theta) \right|_0^{2\pi}$$

$$= \boxed{-\pi}$$



$$\iint_{S_2} F \cdot dS = \iint_D \langle 0, y, -1 \rangle \cdot \langle 0, 0, -1 \rangle \, dA \quad (\text{on } S_2 \ z = 1.)$$

$$= \int_0^{2\pi} \int_0^1 r \, dr \, d\theta$$

$$= \boxed{\pi} \qquad \therefore \quad \boxed{\iint_S F \cdot dS = \pi - \pi = 0.}$$

E163 Let $\vec{F}$ be an inverse square field, use physics notation, $\vec{F} = \frac{c}{r^3}\vec{r}$ where $\vec{r} = \langle x, y, z \rangle$. Show that the flux of $\vec{F}$ across a sphere centered at the origin is independent of the radius R of the sphere. (this is §13.6 #43)

$$\Phi_F = \iint_{S_R} \vec{F} \cdot d\vec{A}$$

$$= \iint_{S_R} \left(\frac{c}{R^2}\hat{r}\right) \cdot \left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right) \qquad \ni \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \phi \leq 2\pi \end{array} \quad \text{and } r = R \text{ on } S_R.$$

$$= \int_0^{2\pi} \int_0^{\pi} c \sin\theta \, d\theta \, d\phi$$

$$= c \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$$

$$= \boxed{4\pi c = \Phi_F}$$

the flux is independent of the radius for this special type of field.