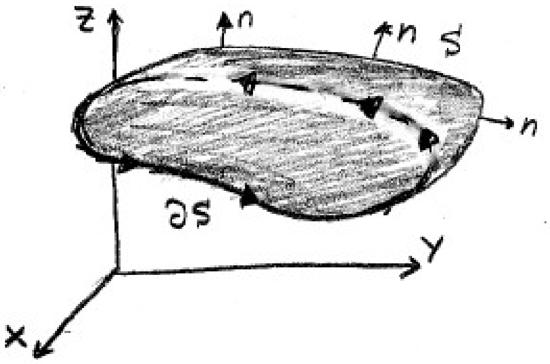
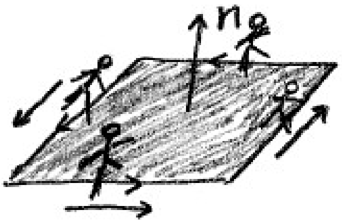
Stokes' Theorem



S is an oriented surface

∂S is boundary with orientation induced from S.

- If we walk around ∂S in the positive direction with our head on the normal's side then S will be on our left always.



- you can also use the right hand rule. Point your thumb along n then your fingers indicate the direction of ∂S.

Th^m (Stokes') Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve ∂S with positive induced orientation from S. Let F be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

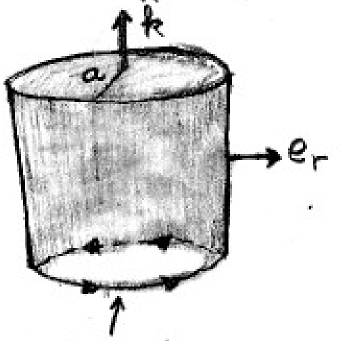
$$\int_{\partial S} F \cdot dr = \iint_S (\nabla \times F) \cdot dS$$

Proof: See Stewart.

$\boxed{E164}$ Notice that $\vec{A} = -\frac{\mu_0 I}{2\pi} \ln(r)\,\hat{k}$ has $A_z = \frac{-\mu_0 I}{2\pi} \ln(r)$

$$\nabla \times \vec{A} = -\frac{\partial A_z}{\partial r} e_\theta = \frac{\mu_0 I}{2\pi} \frac{\partial}{\partial r}\left[\ln(r)\right] e_\theta = \frac{\mu_0 I}{2\pi r} e_\theta = \vec{B} \quad \text{from } \boxed{E161}.$$

where I have used the Th^m from (380) to compute the curl in cylindricals. Its clear that if we let S be the cylinder without the base, then from the calculation in $\boxed{E161}$ we have $\iint_S \vec{B} \cdot d\vec{A} = 0$. Lets check if Stoke's Th^m agrees.



∂S with induced orientation
notice $d\vec{l} = (a\,d\theta)e_\theta$

$$\iint_S \vec{B} \cdot d\vec{A} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{A}$$

$$= \int_{\partial S} \vec{A} \cdot d\vec{l}$$

$$= \int_{\partial S} \frac{-\mu_0 I}{2\pi} \ln(r)\, \hat{k} \cdot (a\,d\theta\, e_\theta) = 0$$

orthogonal.

Remark: The function $\vec{A}$ that gives $\vec{B}$ by $\vec{B} = \nabla \times \vec{A}$ is called the "vector potential". Its the analogue of the scalar potential V that gives $\vec{E}$ by $\vec{E} = -\nabla V$. Actually in general you need both to get $\vec{E}$ and $\vec{B}$... (not req$^{\underline{d}}$ topic)

E165 GREENE'S THEOREM: (§13.4)

An application of Stokes' Th$^{\underline{m}}$ to the xy-plane was discovered before Stokes' Th$^{\underline{m}}$ historically. However, we may view it as an elementary application of Stokes' Th$^{\underline{m}}$ in the following case. Let D be a region bounded by C a simple closed curve, that is let D be a simply connected region with boundary ∂D and suppose D lies in xy-plane. Consider continous vector field $F = \langle P, Q, 0 \rangle$ with continuous partial derivatives of P & Q,

$$\iint_D (\nabla \times F) \cdot dS = \int_{\partial D} F \cdot dr$$

here D has $dS = dx\,dy\,\hat{k}$ and ∂D is oriented counter-clockwise, then since $(\nabla \times F) \cdot \hat{k} = \partial_x Q - \partial_y P$ we find (recall Remark on 390)

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy}$$

In the context of $\mathbb{R}^2$ this applies to $F = \langle P, Q \rangle$.

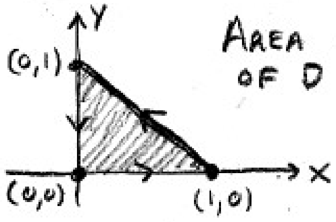
E166 Lets apply Green's Th$^{\underline{m}}$ to the triangular region D formed by the points $(0,0)$, $(1,0)$, $(0,1)$. Let C be the boundary of the triangle oriented CCW, calculate the line integral via Green's Th$^{\underline{m}}$,

$$\int_C -y\,dx + x\,dy = \iint_D \left(\tfrac{\partial}{\partial x}[x] - \tfrac{\partial}{\partial y}[-y] \right) dA \quad : \quad \underline{F = \langle -y, x \rangle \text{, use Green's Th}^{\underline{m}}}$$

$$= \iint 2\,dA$$

$$= 2\,\text{Area}(D)$$

$$= \boxed{1}$$



AREA OF D $= \dfrac{1}{2}$

- We see Green's Th$^{\underline{m}}$ allows us to trade a line integral for a surface integral. This is advantageous if one or the other is easy to calculate, like in the example above.

E167 If we choose a vector field $F = \langle P, Q\rangle$ such that $\partial_x Q - \partial_y P = 1$ then this will give us another method to calculate the area of D.

$$\text{Area}_D = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

So are there such vector fields? Yes, for example

$$F = \langle 0, x\rangle \quad \text{or} \quad F = \langle -y, 0\rangle \quad \text{or} \quad F = \tfrac{1}{2}\langle -y, x\rangle$$

This gives us the following sneaky formulas for area,

$$\boxed{\text{Area}(D) = \int_{\partial D} x\,dy = \int_{\partial D} -y\,dy = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx} \quad (*)$$

Then consider the ellipse $D: x^2/a^2 + y^2/b^2 \le 1$. Then ∂D has parametrization for $0 \le \theta \le 2\pi$

$$x = a\cos\theta \Rightarrow dx = -a\sin\theta\, d\theta$$
$$y = b\sin\theta \Rightarrow dy = b\cos\theta\, d\theta$$

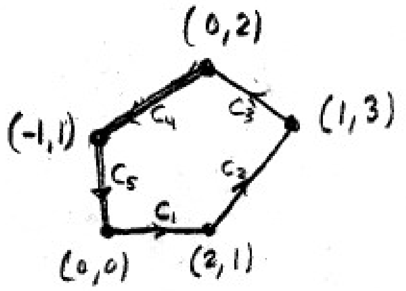
Thus using $(*)$,

$$\begin{aligned}\text{Area}(D) &= \int_{\partial D} x\,dy \\ &= \int_0^{2\pi} (a\cos\theta)(b\cos\theta\, d\theta) \\ &= ab\int_0^{2\pi} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= ab\,\frac{\theta}{2}\Big|_0^{2\pi} + \cancel{\frac{ab}{4}\sin(2\theta)\Big|_0^{2\pi}} \to 0 \\ &= \boxed{\pi ab}\end{aligned}$$

E168 Let C be a line segment from (x_1, y_1) to (x_2, y_2) then we may parametrize $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$ $(0 \le t \le 1)$

$$\begin{aligned}\int_C x\,dy - y\,dx &= \int_0^1 (x_1 + t(x_2 - x_1))[(y_2 - y_1)\,dt] - [y_1 + t(y_2 - y_1)][(x_2 - x_1)\,dt] \\ &= \int_0^1 [x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[\cancel{(x_2 - x_1)(y_2 - y_1)} - \cancel{(y_2 - y_1)(x_2 - x_1)}]]\,dt \\ &= \int_0^1 [x_1 y_2 - \cancel{x_1 y_1} - y_1 x_2 + \cancel{y_1 x_1}]\,dt \\ &= x_1 y_2 - y_1 x_2.\end{aligned}$$

$\boxed{\text{E168 Continued}}$ We found a simple formula for $\int_C x\,dy - y\,dx$ in the case C is a line segment, this coupled to Green's Th$^{\underline{m}}$ gives a simple method to calculate the area of pentagons, octogons, etc... Consider the pentagon P,



$$\int_{C_1} x\,dy - y\,dx = x_1 y_2 - x_2 y_1 = 0$$

$$\int_{C_2} x\,dy - y\,dx = 2(3) - (1)(1) = 5$$

$$\int_{C_3} x\,dy - y\,dx = 2 - 0 = 2$$

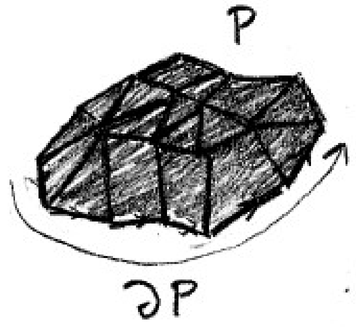
$$\int_{C_4} x\,dy - y\,dx = 0(-1) - (-1)(2) = 2$$

$$\int_{C_5} x\,dy - y\,dx = -1(0) - (0)(1) = 0$$

Then notice that P has boundary $\partial P = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$.

$$\begin{aligned} \text{Area}(P) &= \frac{1}{2}\int_{\partial P} x\,dy - y\,dx \quad \text{using } (*) \text{ on } 414 \\ &= \frac{1}{2}\left[0 + 5 + 2 + 2 + 0\right] \\ &= \boxed{9/2} \end{aligned}$$

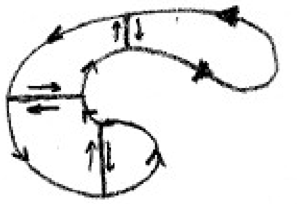
<u>Remark</u>: this is interesting in the context of the xy-plane and Green's Th$^{\underline{m}}$, but this trick will also allow you to calculate the surface area of rather ugly polyhedra.

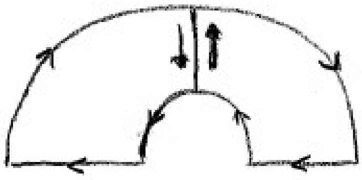


The surface integral of $\nabla \times F$ over P can be calculated from a line integral of F around ∂P. If ∂P lies in the xy-plane we can use an approach similiar to that of $\boxed{\text{E168}}$

REGIONS WITH HOLES, HOW TO APPLY GREEN'S $Th^{\underline{m}}$

According to Stewart TYPE I and TYPE II regions are <u>simple regions</u>. Green's $Th^{\underline{m}}$ can be extended to other regions, lets draw a few pictures to illustrate

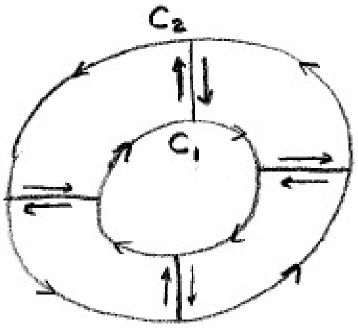




Just like simple regions

$$\boxed{\iint_S (\partial_x Q - \partial_y P)\, dA = \int_{\partial S} P\,dx + Q\,dy}$$

Just divide the shape up into simple regions and add the results together, the places where the divisions share a side do not contribute because the sides have opposing orientations and as such cancel out, thus only the outer edge of the shape matters, or in the case of a donut the inner edge also matters and it must be given CW orientation in contrast to the CCW orientation of the outer edge. Notice each sub region has a CCW orientation.

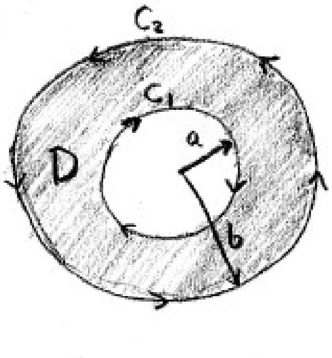


$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy}$$

For regions like this donut we must take care to orient the inner loops in a CW fashion.

<u>Remark</u>: if my pictures here aren't convincing take a look at § 13.4 of Stewart.

$\boxed{E169}$



$C_1 : \begin{cases} x = a\cos t \\ y = -a\sin t \end{cases}$ $\quad C_2 : \begin{cases} x = b\cos t \\ y = b\sin t \end{cases}$ $\quad 0 \leq t \leq 2\pi$

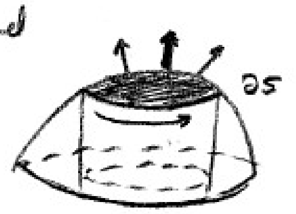
Choose $F = \langle 0, x \rangle$ so $\partial_x Q - \partial_y P = 1$ thus

$$A = \iint_D dA = \int_0^{2\pi} (a\cos t)(-a\cos t\, dt) + \int_0^{2\pi} (b\cos t)(b\cos t\, dt)$$

$$= \int_0^{2\pi} (b^2 - a^2)\tfrac{1}{2}(1 + \cos 2t)\, dt = \boxed{\pi b^2 - \pi a^2}$$

$\boxed{\text{E170}}$ Let $F = \langle yz, xz, xy \rangle$ and let S be the $(z>0)$ surface $x^2+y^2+z^2=4$ bounded by $x^2+y^2=1$. Compute the integral $\iint_S (\nabla \times F) \cdot dS$. Use Stoke's th$^{\underline{m}}$ and we'll not even need to take the curl of F, instead

$$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot dr$$



The boundary of S is ∂S, it satisfies

$$x^2+y^2=1 \quad \text{and} \quad x^2+y^2+z^2=4$$
$$\Rightarrow \quad 1+z^2=4$$
$$\Rightarrow \quad z = \pm\sqrt{3}$$
$$\Rightarrow \quad z = \sqrt{3}$$ since we assume $z>0$ in the statement of the problem.

We parametrize ∂S by $0 \le \theta \le 2\pi$,

$$x = \cos\theta \qquad y = \sin\theta \qquad z = \sqrt{3}$$
$$dx = -\sin\theta\, d\theta \qquad dy = \cos\theta\, d\theta \qquad dz = 0$$

$$\begin{aligned}
\therefore \int_{\partial S} F \cdot dr &= \int_{\partial S} F_1\,dx + F_2\,dy + F_3\,dz \\
&= \int_0^{2\pi} (\sqrt{3}\sin\theta)(-\sin\theta\,d\theta) + (\sqrt{3}\cos\theta)(\cos\theta\,d\theta) + (\sin\theta\cos\theta)(0) \\
&= \int_0^{2\pi} \sqrt{3}\,(\cos\theta\cos\theta - \sin\theta\sin\theta)\,d\theta \\
&= \int_0^{2\pi} \sqrt{3}\,\cos(2\theta)\,d\theta \\
&= \frac{\sqrt{3}}{2}\sin 2\theta \Big|_0^{2\pi} \\
&= 0.
\end{aligned}$$

$\therefore \boxed{\iint_S (\nabla \times F) \cdot dS = 0}$

$\boxed{E171}$ Let S be the rectangle on $z = y$ plane that projects to $0 \leq x \leq 1$, $0 \leq y \leq 3$. Suppose

$$F = \langle x^2, 4xy^3, y^2x \rangle$$

(*) calculate $\int_{\partial S} F \cdot dr$. I'll use Stoke's Th$^{\underline{m}}$ to convert to surface integral

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 4xy^3 & y^2x \end{vmatrix} = \langle 2yx, -y^2, 4y^3 \rangle$$

We'll need to parametrize S in order to compute the surface integral. May I recomend x & y as parameters,

$$r(x,y) = \langle x, y, y \rangle \quad , \quad \underline{0 \leq x \leq 1, \ 0 \leq y \leq 3} \text{ call this } D.$$

$$r_x \times r_y = \langle 1, 0, 0 \rangle \times \langle 0, 1, 1 \rangle = \hat{k} - \hat{j} \quad \left(\begin{matrix}\text{not surprising, think} \\ \text{about } z - y = 0\end{matrix}\right)$$

$$\therefore N(x,y) = \langle 0, -1, 1 \rangle$$

Thus,

$$\int_{\partial S} F \cdot dr = \iint_S (\nabla \times F) \cdot dS \quad , \quad \text{note } dS = \langle 0, -1, 1 \rangle dx\, dy$$

$$= \iint_D \langle 2yx, -y^2, 4y^3 \rangle \cdot \langle 0, -1, 1 \rangle dA$$

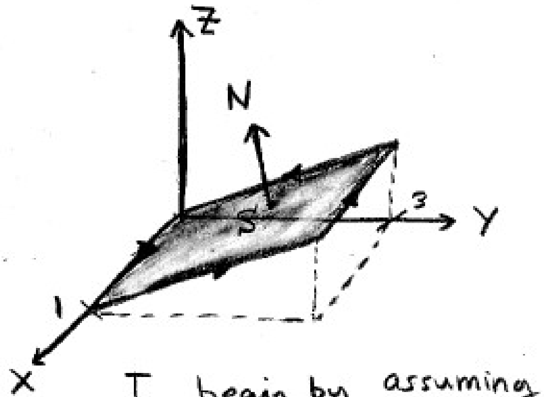
$$= \int_0^3 \int_0^1 (y^2 + 4y^3)\, dx\, dy$$

$$= \int_0^3 (y^2 + 4y^3)\, dy$$

$$= \left(\tfrac{1}{3}y^3 + y^4\right)\Big|_0^3$$

$$= \tfrac{1}{3}(27) + 81$$

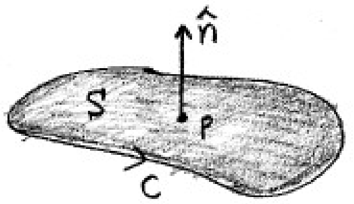
$$= \boxed{90}$$



I begin by assuming ∂S is oriented as pictured. Then to be consistent we must choose N as pictured.

Remark: this example is borrowed from Howard Anton's "CALCULUS" 5^{th} Ed. See $\boxed{E2}$ pg. 978. This is a nice book, lots of history.

Remark: Other texts actually define the curl by a limiting integral condition. This allows an alternative method of calculating the curl in curve linear coordinates.



$$\hat{n}\cdot(\nabla\times F)(P) = \lim_{A\to 0}\frac{1}{A}\oint_C F\cdot dr$$

gives the component of $\nabla\times F$ along $\hat{n}$.

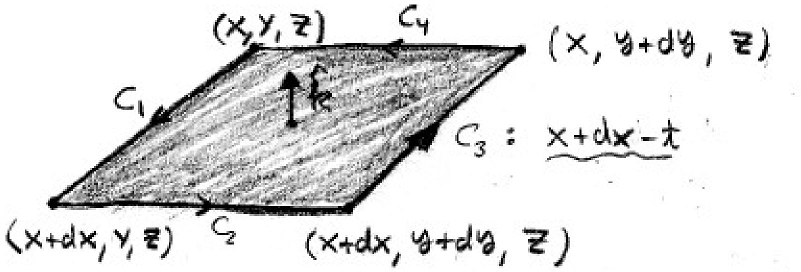
For an infinitesimal area $d\vec{A}$ we can use Stoke's Th$^{\underline{m}}$, $d\vec{A} = (dA)\hat{n}$

$$\iint_{d\vec{A}}(\nabla\times F)\cdot ds = \oint_{\partial(d\vec{A})} F\cdot dr$$

$$\|$$

$$(\nabla\times F)\cdot d\vec{A} = \oint_{\partial(d\vec{A})} F\cdot dr \Rightarrow (\nabla\times F)\cdot\hat{n} = \frac{1}{dA}\oint_{\partial(d\vec{A})} F\cdot dr$$

Let $F = \langle P, Q, R\rangle$ and $d\vec{A} = dx\,dy\,\hat{k}$.



$$\partial(d\vec{A}) = C_1\cup C_2\cup C_3\cup C_4$$
C_1 : $dx = dt$, $dy = dz = 0$
C_2 : $dy = dt$, $dx = dz = 0$
C_3 : $dx = -dt$, $dy = dz = 0$
C_4 : $dy = -dt$, $dx = dz = 0$

$$(\nabla\times F)_z dx\,dy = \oint_{\partial(d\vec{A})} F = \int_{\partial(d\vec{A})} P\,dx + Q\,dy + R\,dz = [\underbrace{Q(x+dx,y,z)}_{(C_2)} - \underbrace{Q(x,y,z)}_{(C_4)}]dy - [\underbrace{P(x,y+dy,z)}_{(C_3)} - \underbrace{P(x,y,z)}_{(C_1)}]dx$$

$$\Rightarrow (\nabla\times F)\cdot\hat{k} = \frac{Q(x+dx,y,z)-Q(x,y,z)}{dx} - \frac{P(x,y+dy,z)-P(x,y,z)}{dy}$$

$$\boxed{(\nabla\times F)_3 = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}$$

Remark: Notice we didn't assume that $\nabla\times F$ had this form, we derived it here. Personally I'm not much of a fan of this approach, if you'd like to see more check out problems #23 & 24 of §7.3 Colley 1$^{\underline{st}}$ Ed.

Conservative Vector Fields Revisited Once More

We have seen the physical and mathematical significance of the potential function, path independence etc... Lets make a list of facts. The following are equivalent, assuming dom(F) is simply connected.

(1) F is conservative

(2) $\exists f$ such that $F = \nabla f$

(3) F is path independent, $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for all paths C_1 & C_2 with same initial & terminal points

(4) dom(F) simply connected & $\nabla \times F = 0$

(5) $\oint_C F \cdot dr = 0$ for all closed paths C.

If we drop the demand of dom(F) being simply connected then we'll not be able to assume $\nabla \times F = 0 \Rightarrow F = \nabla f$. Lets see how Stoke's Th$^{\text{m}}$ connects these statements. Assume dom(F) is simply connected, consider closed path $C = \partial S$ where S is some surface that takes C as its consistently oriented boundary.

$$\iint_S (\nabla \times F) \cdot dS = \oint_C F \cdot dr$$

If (5) holds then $\iint_S (\nabla \times F) \cdot dS = 0$ for all surfaces which implies $\nabla \times F = 0$, thus $(5) \Rightarrow (4)$. The other implications we've argued earlier. Notice that we need dom(F) to be simply connect in order that $\oint_C F \cdot dr$ doesn't get caught on any holes, we wouldn't have $\iint_S (\nabla \times F) \cdot dS = 0$ $\forall$ surfaces around a point in dom(F), we'd have to worry about the holes in dom(F) and ultimately that spoils the implication $\nabla \times F = 0 \Rightarrow F$ conservative.

(F conservative, $\nabla f = F \Rightarrow \nabla \times F = 0$ is always true)

Divergence Theorem

A simple solid region is one which can be encapsulated by a sphere, ellipsoid, cube etc... Its a bounded subset of $\mathbb{R}^3$, with no holes.

$\boxed{\text{Th}^{\underline{m}}}$ Let E be a simple solid region with $S = \partial E$ the boundary given an outward orientation. Let F be a vector field whose component functions have continuous partials on E. Then

$$\iint_{\partial E} F \cdot ds = \iiint_E (\nabla \cdot F)\, dV$$

<u>Proof</u>: See Stewart §13.8, its geometry plus the FTC as usual.

$\boxed{E172}$ Find flux $\Phi_F = \iint_S F \cdot ds$ over the sphere $S: x^2+y^2+z^2 = R^2$ where $F = \langle x, y, z \rangle$. This example (Stolen from Stewart p.968) is tailor made for the divergence Th$^{\underline{m}}$. Notice

$$\nabla \cdot F = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\iint_S F \cdot ds = \iiint_{\underbrace{x^2+y^2+z^2 \leq R^2}_{E}} (\nabla \cdot F)\, dV = 3\iiint_E dV = 3 \cdot \frac{4}{3}\pi R^2 = \boxed{4\pi R^2 = \Phi_F}$$

$\boxed{E173}$ Let us consider the sphere of radius R again find flux of $F = \langle xy^2, yz^2, zx^2 \rangle$. Note $\nabla \cdot F = y^2 + z^2 + x^2 = \rho^2$

$$\Phi_F = \iint_S F \cdot ds = \iiint_E \rho^2\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \rho \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^R \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi$$

$$= \frac{1}{4}R^4 \cdot 2\pi \cdot 2$$

$$= \boxed{\pi R^4 = \Phi_F}$$

$\boxed{E174}$ Consider $E = \frac{kQ}{r^2}\hat{r}$ in the physics notation, so $r^2 = x^2+y^2+z^2$ and $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$, $s^2 = x^2+y^2$.

$$\nabla \cdot \left(\frac{kQ}{r^2}\hat{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 E_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \underbrace{E_\theta}_{0}\right] + \frac{1}{r\sin\theta}\underbrace{\frac{\partial E_\phi}{\partial\phi}}_{0}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{kQ}{r^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[kQ\right]$$

$$= 0. \quad \therefore \quad \underline{\nabla \cdot E = 0}$$

Then by the divergence $Th^{\underline{m}}$ we calculate the flux of E through the sphere $x^2+y^2+z^2 = R^2$,

$$\iint_S E \cdot dS = \iiint (\nabla \cdot E)\, dV = 0 \Rightarrow \underline{\Phi_E = 0} \quad (*)$$

Lets check this against explicit calculation of the surface integral, note $dS = R^2 \sin\theta \, d\theta d\phi \, \hat{r}$ thus

$$\iint_S E \cdot dS = \int_0^{2\pi}\int_0^{\pi} \frac{kQ}{R^2} \cdot R^2 \sin\theta \, d\theta \, d\phi \qquad \left(\text{recall } k = \frac{1}{4\pi\epsilon_0}\right)$$

$$= kQ\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta = 4\pi k Q = \frac{4\pi}{4\pi\epsilon_0}Q = \frac{Q}{\epsilon_0}$$

So which is it $\Phi_E = 0$ or $\Phi_E = Q/\epsilon_0$?

<u>RESOLUTION TO PARADOX</u>: $\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r})$ where $\delta(\vec{r})$ is the 3-d Dirac Delta function, $\int f(\vec{r})\,\delta(\vec{r})\,dV = f(0)$. In simple terms $\delta(\vec{r}) = 0$ everywhere except at $\vec{r} = 0$ where it is infinite. Then $\nabla \cdot E = \frac{Q}{\epsilon_0}\delta(\vec{r})$, then one of Maxwell's $Eq^{\underline{n}}$'s is $\nabla \cdot E = \rho/\epsilon_0$ thus for a point charge Q centered at the origin the charge density is $\rho(\vec{r}) = Q\,\delta(\vec{r})$. Mathematically our $sol^{\underline{n}}$ was bogus since $\text{dom}(E) \not\ni (0,0,0)$. It had a hole. You can look at p. 969–970 to see how Stewart dodges this.

Remark: You'll see $\delta(x)$ in ma 341 when you study discontinuous forcing functions on springs and things. the $\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$ where $\int f(x)\,\delta(x)\,dx = f(0)$. These Dirac Delta functions turn integration into evaluation. The mathematics to seriously do these things wasn't known until the early 20^{th} century, see the work of Schwartz. If you object to point charges you could insist that the charge Q was smeared out over some tiny sphere that would give a density of:

$$\rho_1 = \begin{cases} \dfrac{Q}{4/3\,\pi a^3}\,\frac{4}{3}\pi r^3 & 0 \leq r \leq a \\ 0 & r > a \end{cases}$$

 Q

$$\rho_2 = Q\,\delta(\vec{r})$$ (this picture is way to big, its a point) · Q

The interesting thing is that for $r > a$ both ρ_1 & ρ_2 yield the same field. This is Gauss Law,

$$\frac{Q_{enc}}{\epsilon_0} = \underbrace{\iint_{\partial S} E \cdot dA = \iiint_S (\nabla \cdot E)\,dV}_{\text{divergence Th}^{\underline{m}}} = \underbrace{\iiint \rho_1/\epsilon_0\,dV}_{\nabla \cdot E = \rho/\epsilon_0} = \iiint \rho_2/\epsilon_0\,dV$$

$\nabla \cdot E = \rho/\epsilon_0$ Gauss' Law in differential form

By symmetry $\int E \cdot dA = 4\pi R^2 |E|$ and $E = E_r \hat{r}$, $E_\theta = E_\phi = 0$. thus

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{R^2}\hat{r}}$$

Remark: the Divergence Th$^{\underline{m}}$ is also called Gauss Th$^{\underline{m}}$. The calculation sketched above connects the so-called integral & differential formulations of Gauss' Law

$$\Phi_E = Q/\epsilon_0 \iff \nabla \cdot E = \rho/\epsilon_0$$

Calculus III and Differential Forms

I'll give a brief advertisement here. You can look in my ma 430 notes or Colley for more details. In short differential forms unify the concepts of calculus III in a slick elegant fashion. <u>Fundamental Correspodance</u>:

$$F = \langle P, Q, R \rangle \quad \begin{array}{l} \nearrow \ \omega_F = P\,dx + Q\,dy + R\,dz \\ \searrow \ \Phi_F = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy \end{array}$$

we say ω_F is the one-form corresponding to F while Φ_F is the two-form corresponding to F, $\underline{\omega_A \wedge \omega_B = \Phi_{A \times B}}$

the wedge product generalizes the cross product.

<u>Exterior Derivative</u>:

usual	corresponds	Output	Corresponds to
f	function f	df	$\omega_{\nabla f} = df$
F	ω_F	$d\omega_F$	$\Phi_{\nabla \times F} = d\omega_F$
G	Φ_G	$d\Phi_G$	$d\Phi_G = (\nabla \cdot G)\,dx \wedge dy \wedge dz$

the single operation of exterior differentiation reproduces the gradient, curl and divergence!

<u>Integration of Forms</u>:

$$\int_C F \cdot dr = \int_C \omega_F$$

$$\iint_S F \cdot dS = \int_S \Phi_F$$

$$\iiint_E f\,dV = \int_E f\,dx \wedge dy \wedge dz$$

- Notice a p-form can only be integrated over a p-dim'l space. In contrast we integrate vector fields along a line or accross a surface.

GENERALIZED STOKE'S THEOREM

$$\boxed{\int_M d\beta = \int_{\partial M} \beta}$$

this encodes the FTC, GREEN'S, STOKES and DIVERGENCE Th$^{\underline{m}}$ in one unified frame work.

$$\omega_{\nabla f} = df \longrightarrow$$

$$\int_C F \cdot dr = \int_C \omega_F \longrightarrow$$

$$\int_C df = \int_{\partial C = \{a,b\}} f. \longrightarrow$$

$$\boxed{\int_C \nabla f \cdot dr = f(b) - f(a)}$$

FTC for line integrals

$$d\omega_F = \Phi_{\nabla \times F}$$

$$\iint_S F \cdot dS = \int_S \Phi_F$$

$$\int_S d\omega_F = \int_{\partial S} \omega_F$$

$$\int_S \Phi_{\nabla \times F} = \int_S d\omega_F = \int_{\partial S} \omega_F$$

$$\boxed{\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot dr}$$

Stoke's Th$^{\underline{m}}$

$$d\Phi_F = (\nabla \cdot F)\, dx \wedge dy \wedge dz$$

$$\iiint_E f\, dV = \int_E f\, dx \wedge dy \wedge dz$$

$$\int_E d\Phi_F = \int_{\partial E} \Phi_F$$

$$\boxed{\iiint_E (\nabla \cdot F)\, dV = \iint_{\partial E} F \cdot dS}$$

Divergence or Gauss' Th$^{\underline{m}}$.

the true beauty of the theory of differential forms is that they unify the 3-d calculus and better yet are defined in higher dimensions. Differential forms replace vector fields as the object of primary physical interest in modern physical theory.