Differential Forms in $\mathbb{R}^3$, Bonus Project for Ma 242-011, due May 7

The purpose of this bonus project is to introduce you to differential forms. We will take a slightly informal approach, I will give you greedy definitions rather than minimal definitions. This project is meant to be self-contained, you should be able to use the properties I list to do the problems. You are free to work with each other and ask individuals outside the course. Of course ask me if it is unclear what I want. This project is worth 10 bonus points, there is nothing to difficult here, its really just algebra.

wedge product, the algebra of differential forms

Lets begin with the **wedge product**. Suppose that α, β are differential forms of degrees p and q respectively then by definition the wedge product of α and β gives a new differential form of degree $p+q$ which we denote $\alpha \wedge \beta$. Given differential forms α, β, γ and $a, b \in \mathbb{R}$ then

$$\begin{array}{ll} (i.) & \alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma \\ (ii.) & \alpha \wedge (a\beta + b\gamma) = a\alpha \wedge \beta + b\alpha \wedge \gamma \\ (iii.) & \alpha \wedge (a\beta) = a\alpha \wedge \beta \\ (iv.) & \alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma \\ (v.) & \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \end{array} \tag{1}$$

The properties $i., ii., iii.$ simply say that the wedge product distributes across addition of real numbers and differential forms, property $iv.$ says that the wedge product is associative and the most interesting property is $v.$ which we will see leads to some interesting results much like the cross product.

On $\mathbb{R}^3$ there are four nontrivial types of differential forms, they all obey the algebraic rules $i. - v.$,

Name	general example	degree p
zero-forms	f	p=0
one-forms	$\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$	$p = 1$
two-forms	$\beta = \beta_1 dy \wedge dz + \alpha_2 dz \wedge dx + \alpha_3 dx \wedge dy$	$p = 2$
three-forms	$\gamma = g dx \wedge dy \wedge dz$	$p = 3$

where $f, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, g$ are all functions of x, y, z. Now I'll let you work out a few examples,

Problem 1

Prove that $dx \wedge dx = 0, dy \wedge dy = 0, dz \wedge dz = 0$. These follow from property v. Note that the degrees of dx, dy, dz are $p = 1$.

Problem 2

Suppose that $\alpha = dx + 3dy$ and $\beta = 3dx + 5dz$. Calculate, $\alpha \wedge \beta$, $\beta \wedge \alpha$, $\alpha \wedge \alpha$, and $(3\alpha + \beta) \wedge dz$. You will need the previous problem's result to simplify your answers.

differential calculus of differential forms, the exterior derivative

Next we define a differentiation called the "exterior derivative", assume that f, α, β, γ are still defined as in the table above,

$$\begin{array}{ll} (i.) & df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ (ii.) & d\alpha = d\alpha_1 \wedge dx + d\alpha_2 \wedge dy + d\alpha_3 \wedge dz \\ (iii.) & d\beta = d\beta_1 \wedge dy \wedge dz + d\beta_2 \wedge dz \wedge dx + d\beta_3 \wedge dx \wedge dy \\ (iv.) & d\gamma = dg \wedge dx \wedge dy \wedge dz \end{array} \tag{2}$$

where $ii., iii., iv.$ all use $i.$ to define $d\alpha_1, d\alpha_2, d\alpha_3, d\beta_1, d\beta_2, d\beta_3, dg$. These calculations look the same as taking total differentials, the extra wrinkle is that we wedge together the differentials. If you want to be picky then we should use different symbols to distinguish between the dx from our course and the dx we wedge now. However, we'll not make that distinction. You can just think of them as the same. Notice that the exterior derivative takes a p-form α to a $p+1$-form $d\alpha$.

Problem 3

Let $f = xyz$. Calculate df, $df \wedge dx$, and $d(df)$.

Problem 4

Let $\alpha = ydx + zdy + xdz$ calculate $d\alpha$ and $d(d\alpha) = 0$.

Problem 5

Let $\gamma = gdx \wedge dy \wedge dz$ prove that $d\gamma = 0$.

Problem 6

Prove that $d(d\alpha) = 0$ for any $p - form$. You need to check the cases $p = 0, 1, 2$, the case $p = 3$ you did in the previous problem.

dictionary between vector calculus and differential forms

We now explore the connection between differential forms and vector fields. Given a vector field $A = (a, b, c)$ we define the work-form mapping and flux-form mapping,

$$A \mapsto \omega_A = adx + bdy + cdz$$

$$A \mapsto \Phi_A = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$$

this means we can connect two different differential forms to a single vector field. I'll let you verify a few properties of these mappings to get familiar with them. These are not hard, if you get stuck ask.

Problem 7

Let F and G be vector fields and let $c \in \mathbb{R}$ then show that,

$$\omega_{F+G} = \omega_F + \omega_G$$

$$\omega_{cF} = c\omega_F$$

$$\Phi_{F+G} = \Phi_F + \Phi_G$$

$$\Phi_{cF} = c\Phi_F$$

$$\Phi_{\hat{i}} = dy \wedge dz, \quad \Phi_{\hat{j}} = dz \wedge dx, \quad \Phi_{\hat{k}} = dx \wedge dy$$

$$\omega_F \wedge \omega_G = \Phi_{F \times G}$$

where $F \times G$ denotes the cross-product of F with G.

Problem 8

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$ be smooth vector fields on $\mathbb{R}^3$. **Show**
(i.) $df = \omega_{\nabla f}$
(ii.) $d\Phi_G = (\nabla \cdot G) dx \wedge dy \wedge dz$
(iii.) $d\omega_F = \Phi_{\nabla \times F}$

Problem 9

What vector identities are encoded by the equation $d(d\alpha) = 0$. Use the correspondances found in the last problem to guide you.

Generalized Stokes Theorem, integral calculus of differential forms

Integrals of differential forms are defined in terms of ordinary integrals. A degree p-form can only be integrated over a p-dimensional space. That is a one-form integates over a curve, a two-form over a surface, a three-form over a volume and a zero-form over a zero-dimensional set. In particular,

$$\int_C \omega_F = \int_C F \cdot dl \qquad \int_S \Phi_G = \int_S G \cdot dA \qquad \int_V f dx \wedge dy \wedge dz = \int \int \int_V f dx dy dz$$

And for a function f which is a zero-form we can integrate over a discrete set $\{a, b\} = \partial C$ which is the boundary of a curve C, that is a, b are the endpoints.

$$\int_{\{a,b\}} f = f(b) - f(a)$$

The interesting thing about integrating differential forms is that the Stoke's Theorem generalizes to the Generalized Stoke's Theorem,

$$\int_M d\alpha = \int_{\partial M} \alpha$$

where M is some space and ∂M is its boundary. This theorem encompasses just about every theorem of integral calculus we have dicussed in Ma 242. I'll let you prove that, its not hard basically you just need to use the results of the problem 8 plus the statements given in the paragraph just above that connect line, surface and volume integrals to differential form integrations.

Problem 10

Using the items given above **prove the following statements follow from the Generalized Stoke's Theorem**

(i.) $\int_C (\nabla f) \cdot dl = f(b) - f(a)$

(ii.) $\int_S (\nabla \times F) \cdot dA = \int_{\partial S} F \cdot dl$

(iii.) $\int \int \int_V (\nabla \cdot G) dx dy dz = \int_{\partial V} G \cdot dA$

for a curve C, a surface S and a volume V all consistently oriented subsets of $\mathbb{R}^3$ and we assume that the vector fields are smooth so all the derivatives listed above are well-behaved.

Additional problems are available if you ask.