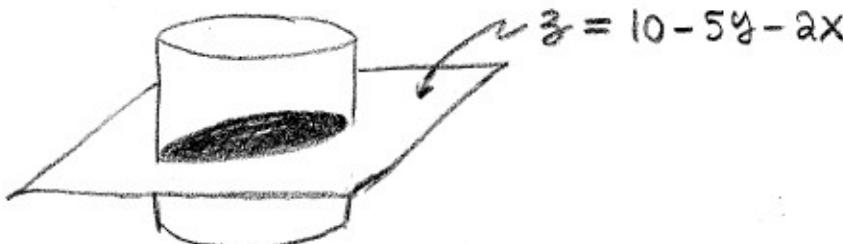


§10.6 #2 Find area of part of plane $2x + 5y + 3z = 10$ inside the cylinder $x^2 + y^2 = 9$.



$$\mathbf{r}(x, y) = \langle x, y, 10 - 5y - 2x \rangle \text{ for } \{x^2 + y^2 \leq 9\} = D \subset \mathbb{R}^2$$

This surface is parametrized by x & y .

$$\mathbf{r}_x = \langle 1, 0, -2 \rangle$$

$$\mathbf{r}_y = \langle 0, 1, -5 \rangle$$

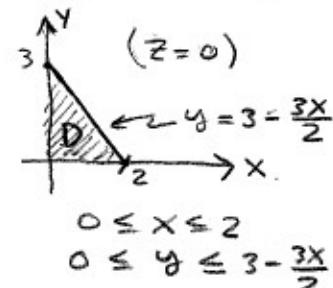
$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{vmatrix} = \langle 2, 5, 1 \rangle \text{ normal to surface (well duh.)}$$

$$A(S) = \iint_D |\mathbf{r}_x \times \mathbf{r}_y| dA = \iint_D \sqrt{30} dA = \sqrt{30} \iint_D dA = \sqrt{30} \cdot \pi(3)^2 = \boxed{9\pi\sqrt{30}}$$

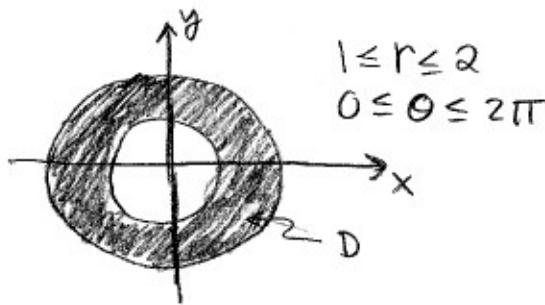
Remark: I have done #2 a kinda-weird way. I mean the easy way is to use formula 6 since this is a $z = f(x, y)$ graph.

§10.6 #3 $z = 6 - 3x - 2y$ with $x, y, z \geq 0$. Note $z=0 \Rightarrow 2y = 6 - 3x \Rightarrow y = 3 - \frac{3}{2}x$

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \int_0^2 \int_0^{3-3x/2} \sqrt{1 + 9 + 4} dy dx \\ &= \int_0^2 \sqrt{14} (3 - 3x/2) dx \\ &= \sqrt{14} \left(x - \frac{1}{4}x^2\right) \Big|_0^2 \\ &= \boxed{3\sqrt{14}} \end{aligned}$$



§12.6 #5 Find surface area of $Z = y^2 - x^2$ between cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2} dA$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2} r dr d\theta$$

(use polar coordinates for this integral, $dA = r dr d\theta$ remember?)

$$= 2\pi \left(\frac{2}{3} \frac{1}{8} (1+4r^2)^{3/2} \right) \Big|_1^2$$

$$= \frac{\pi}{6} ((17)^{3/2} - (5)^{3/2})$$

$$= \boxed{\frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})}$$

I integrated & checked it by differentiating, you may need to do this explicitly to be safe.

§12.6 #7 Surface with parametric eq's $r(u, v) = \langle u^2, uv, \frac{1}{2}v^2 \rangle$
where $0 \leq u \leq 1$ and $0 \leq v \leq 2$. Calculate the tangents

$$r_u = \langle 2u, v, 0 \rangle$$

$$r_v = \langle 0, u, v \rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \langle v^2, -2uv, 2u^2 \rangle$$

We note $|r_u \times r_v| = \sqrt{v^4 + 4u^2v^2 + 4u^2} = \sqrt{(v^2 + 2u^2)^2} = v^2 + 2u^2$.

$$A = \iint_0^1 |r_u \times r_v| du dv = \int_0^2 \int_0^1 (v^2 + 2u^2) du dv$$

$$= \int_0^2 (v^2 + \frac{2}{3}) dv$$

$$= \frac{1}{3}(v^3 + 2v) \Big|_0^2 = \frac{1}{3}(8 + 4) = \frac{12}{3} = \boxed{4}$$

§12.6 #22 Consider the parametric eq's

$$x = a \cosh u \cos v$$

$$y = b \cosh u \sin v$$

$$z = c \sinh u$$

We show these parametrize a "hyperboloid of one-sheet". The essential ideas are $\cos^2 v + \sin^2 v = 1$ (hopefully you know!)

and $\cosh^2 u - \sinh^2 u = 1$ (let me show you how)

$$\begin{aligned} \cosh(u) &= \frac{1}{2}(e^u + e^{-u}) \\ \sinh(u) &= \frac{1}{2}(e^u - e^{-u}) \end{aligned} \quad \left. \begin{aligned} \cosh^2 u - \sinh^2 u &= \frac{1}{4}[(e^u + e^{-u})^2 - (e^u - e^{-u})^2] \\ &= \frac{1}{4}[(e^{2u} + 2 + e^{-2u}) - (e^{2u} - 2 + e^{-2u})] \\ &= \frac{1}{4}(2 - (-2)) = \frac{4}{4} = 1 \end{aligned} \right.$$

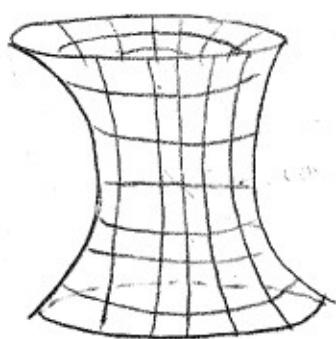
By the way $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ so the analogy between \cos & \cosh and \sin and \sinh is very far-reaching.

Ok with that background in place, we need to get rid of a, b, c somehow, ~~divide~~: ~~divide~~ divide. And consider,

$$\begin{aligned} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 &= \cosh^2 u \underline{\cos^2 v} + \cosh^2 u \underline{\sin^2 v} - \sinh^2 v \\ &= \cosh^2 u (\cos^2 v + \sin^2 v) - \sinh^2 v \\ &= \cosh^2 u - \sinh^2 v \\ &= 1 \end{aligned}$$

this is a hyperboloid of one-sheet (see pg. 682 table 2).

(b) Let $a=1, b=2, c=3$ and plot between $z = \pm 3$



It's an ellipse for each fixed value of z . If we fix $u = u_0$ then let v vary then it traces out an ellipse on the plane

$$z = 3 \sinh u_0$$

with eq:

$$\underbrace{x^2 + \frac{y^2}{4}}_{=} = \sinh^2 u_0 + 1$$

§10.6 #22 continued We wish to set up surface area integral for the specific case plotted in part b. We note that the parameter V is essentially a polar angle and we want $0 \leq V \leq 2\pi$. Now U needs to be chosen so that

$$-3 \leq z \leq 3 \Rightarrow -3 \leq 3 \sinh(U) \leq 3$$

$$\Rightarrow -1 \leq \sinh(U) \leq 1$$

$$\Rightarrow \sinh^{-1}(-1) \leq U \leq \sinh^{-1}(1)$$

$$\Rightarrow -\ln(1+\sqrt{2}) \leq U \leq \ln(1+\sqrt{2})$$

can do since,
 $\frac{d}{du}(\sinh u) = \cosh(u) > 0$
 it's increasing
 function.

$$\text{Notice } \sinh(\ln(1+\sqrt{2})) = \frac{1}{2}(e^{\ln(1+\sqrt{2})} - e^{-\ln(1+\sqrt{2})})$$

$$= \frac{1}{2}(1+\sqrt{2} - \frac{1}{1+\sqrt{2}})$$

$$= \frac{1}{2}\left(\frac{1+2\sqrt{2}+2-1}{1+\sqrt{2}}\right) = \frac{1+\sqrt{2}}{1+\sqrt{2}} = 1$$

$$\text{thus } \sinh^{-1}(1) = \ln(1+\sqrt{2}) \text{ as claimed.}$$

$$\text{Notice that } r(u,v) = \langle \cosh u \cos v, 2\cosh u \sin v, 3 \sinh u \rangle$$

$$r_u = \langle \sinh u \cos v, 2\sinh u \sin v, 3\cosh u \rangle$$

$$r_v = \langle -\cosh u \sin v, 2\cosh u \cos v, 0 \rangle$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sinh u \cos v & 2\sinh u \sin v & 3\cosh u \\ -\cosh u \sin v & 2\cosh u \cos v & 0 \end{vmatrix}$$

$$r_u \times r_v = \langle -6\cosh^2 u \cos v, -3\cosh^2 u \sin v, 2\sinh u \cosh u \rangle$$

Thus,

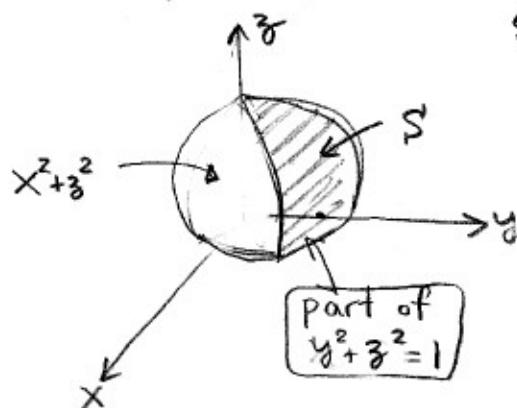
$$A = \iint_{0 \rightarrow \ln(1+\sqrt{2})}^{2\pi} \sqrt{36\cosh^4 u \cos^2 v + 9\cosh^4 u \sin^2 v + 4\sinh^2 u \cosh^2 u} \, du \, dv$$

§12.6 #24 find surface area of the intersection of

H104

$x^2 + z^2 = 1$ and $y^2 + z^2 = 1$. By symmetry it is sufficient to find area of one face of the surface then quadruple it. In my crude sketch I indicate which quarter we'll focus our efforts on.

notice we can parametrize S by



$$\mathbf{r}(x, \theta) = \langle x, \cos \theta, \sin \theta \rangle$$

we must satisfy both eq's on boundary of S .

$$1.) x^2 + z^2 = x^2 + \sin^2 \theta = 1$$

$$2.) y^2 + z^2 = \cos^2 \theta + \sin^2 \theta = 1 \quad (\text{true on all of } S)$$

We see 2.) is automatic while 1.) has more to say,

$$\begin{aligned} x^2 + z^2 \leq 1 &\Leftrightarrow x^2 + \sin^2 \theta \leq 1 \\ &\Leftrightarrow x^2 \leq 1 - \sin^2 \theta = \cos^2 \theta \\ &\Leftrightarrow -\cos \theta \leq x \leq \cos \theta \end{aligned}$$

By the graph $-\pi/2 \leq \theta \leq \pi/2$, notice this traces out the surface from $z = -1$ up to $z = 1$.
 $\mathbf{r}(x, -\pi/2) = (x, \cos(-\pi/2), \sin(-\pi/2)) = \langle x, 0, -1 \rangle$ in-between we have
 $\mathbf{r}(x, \pi/2) = (x, \cos(\pi/2), \sin(\pi/2)) = \langle x, 0, 1 \rangle$ non zero x

for example at $\theta = 0$ get $\mathbf{r}(x, 0) = \langle x, 1, 0 \rangle$
where $-1 \leq x \leq 1$. Anyway,

$$\begin{aligned} \mathbf{r}_x &= \langle 1, 0, 0 \rangle & \mathbf{r}_x \times \mathbf{r}_\theta &= \langle 0, -\cos \theta, -\sin \theta \rangle \\ \mathbf{r}_\theta &= \langle 0, -\sin \theta, \cos \theta \rangle \end{aligned}$$

$$\text{thus } |\mathbf{r}_x \times \mathbf{r}_\theta| = \cos^2 \theta + \sin^2 \theta = 1$$

$$A_S = \int_{-\pi/2}^{\pi/2} \int_{-\cos \theta}^{\cos \theta} dx d\theta = \int_{-\pi/2}^{\pi/2} 2\cos \theta d\theta = 2\sin \theta \Big|_{-\pi/2}^{\pi/2} = 2(1+1) = 4$$

$$\therefore \boxed{\text{total area} = 16}$$

Remark: You could also use x & z as parameters, but that will lead you to an improper integral (that converges thankfully).

§13.6 #5 Let S be the surface with parametric eq's

$$\mathbf{r}(u, v) = \langle u^2, u \sin v, u \cos v \rangle \text{ for } 0 \leq u \leq 1, 0 \leq v \leq \pi/2$$

$$\mathbf{r}_u = \langle 2u, \sin v, \cos v \rangle$$

$$\mathbf{r}_v = \langle 0, u \cos v, -u \sin v \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & \sin v & \cos v \\ 0 & u \cos v & -u \sin v \end{vmatrix} = \langle -u, +2u^2 \sin v, 2u^2 \cos v \rangle$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2 + 4u^4 \sin^2 v + 4u^4 \cos^2 v} = \sqrt{u^2 + 4u^4} = u\sqrt{1+4u^2}$$

Now calculate the surface integral of $f(x, y, z) = yz$

$$\int_S yz dS = \iint_0^{\pi/2} (u \sin v)(u \cos v) u \sqrt{1+4u^2} du dv$$

$$= \int_0^{\pi/2} \sin v \cos v dv \int_0^1 u^3 \sqrt{1+4u^2} du$$

$$= \left(\frac{1}{2} \sin^2(v) \Big|_0^{\pi/2} \right) \left(\int_1^5 \frac{1}{32} (W^{3/2} - W^{1/2}) dW \right)$$

$$= \frac{1}{2} \cdot \frac{1}{32} \left[\frac{2}{5} W^{5/2} - \frac{2}{3} W^{3/2} \Big|_1^5 \right]$$

$$= \frac{1}{32} \left[\left(\frac{1}{5} 5^{5/2} - \frac{1}{3} 5^{3/2} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) \right]$$

$$= \frac{1}{32} \left[\sqrt{5} \left(5 - \frac{5}{3} \right) + \frac{2}{15} \right]$$

$$= \frac{1}{32} \left[\frac{10}{3} \sqrt{5} + \frac{2}{15} \right]$$

$$= \boxed{\frac{5\sqrt{5}}{48} + \frac{1}{240}}$$

(I make a w-substitution.)

$$W = 1 + 4u^2$$

$$dW = 8u du \rightarrow u du = \frac{1}{8} dW$$

$$u^2 = \frac{1}{4}(W-1)$$

$$u^2 \sqrt{1+4u^2} du = \frac{1}{4}(W-1) \sqrt{W} \frac{1}{8} dW$$

$$W(0) = 1$$

$$W(1) = 5.$$

§13.6 #8 S is triangular region with corners $(1, 0, 0), (0, 2, 0), (0, 0, 2)$. In order to take a surface integral over S we must first find a parametric description of it. Here S is a plane with the line segments

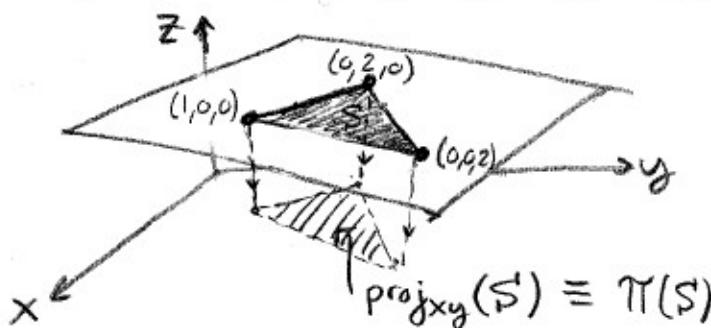
$A = \langle -1, 2, 0 \rangle$ and $B = \langle -1, 0, 2 \rangle$ using (using given points)

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \langle 4, 2, 2 \rangle \Rightarrow \text{normal can be taken as } \langle 2, 1, 1 \rangle.$$

$$S: 2(x-1) + y + z = 0 \therefore \underline{z = 2 - 2x - y}$$

$$\mathbf{r}(x, y) = \langle x, y, 2 - 2x - y \rangle \quad \text{but what values of } x \text{ & } y \text{ give } S?$$

§13.6 #8 Still working out the parametrization. We have the
 $\text{eq}^2 \quad r(x, y) = \langle x, y, 2-2x-y \rangle$ and that will
give us the surface S and a lot more. We just want S' .

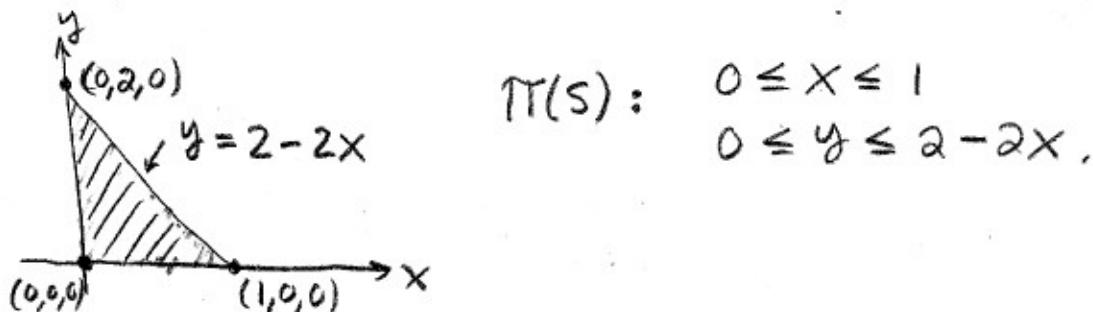


$$\Pi(1, 0, 0) = (1, 0, 0)$$

$$\Pi(0, 2, 0) = (0, 2, 0)$$

$$\Pi(0, 0, 2) = (0, 0, 2)$$

My picture is bogus of course, I'm just trying to get the concept across, two of our points already lie on (xy) -plane to begin with. Any way $\Pi(S)$ is the appropriate parameter space ($r: \Pi(S) \rightarrow S$)



Now we can calculate the desired surface integral,

$$\begin{aligned}
\iint_S xy \, dS &= \int_0^1 \int_0^{2-2x} xy \sqrt{1+4+1} \, dy \, dx \\
&= \int_0^1 \sqrt{6} \frac{x}{2} (2-2x)^2 \, dx \\
&= 2\sqrt{6} \int_0^1 x(1-x)^2 \, dx \\
&= 2\sqrt{6} \int_0^1 (x - 2x^2 + x^3) \, dx \\
&= 2\sqrt{6} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\
&= 2\sqrt{6} \left(\frac{9-8}{12} \right) \\
&= \frac{\sqrt{6}}{6} = \boxed{\frac{1}{\sqrt{6}}}
\end{aligned}$$

§13.6 #12 Let S' be the surface $x = 4 + 2z^2$ for $0 \leq y \leq 1$, $0 \leq z \leq 1$.
The natural choice is $\mathbf{r}(y, z) = \langle y + 2z^2, y, z \rangle$ this parametrizes S' .

$$\mathbf{r}_y = \langle 1, 1, 0 \rangle$$

$$\mathbf{r}_z = \langle 4z, 0, 1 \rangle$$

$$\mathbf{r}_y \times \mathbf{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 4z & 0 & 1 \end{vmatrix} = \langle 1, -1, -4z \rangle$$

$$|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{1+1+16z^2} = \sqrt{2+16z^2}$$

Thus,

$$\begin{aligned} \iint_S 3 dS' &= \int_0^1 \int_0^1 3 \sqrt{2+16z^2} dy dz \\ &= \int_0^1 3 \sqrt{2+16z^2} dz \\ &= \frac{2}{3} \frac{1}{3/2} (2+16z^2)^{3/2} \Big|_0^1 = \frac{1}{48} \left[(18)^{3/2} - (2)^{3/2} \right] = \frac{2^{3/2}}{48} \left[\overbrace{9^{3/2}-1}^{27-1} \right] = \boxed{\frac{13}{12}\sqrt{2}} \end{aligned}$$

§13.6 #15 Let S' be hemisphere $x^2+y^2+z^2=4$, $z \geq 0$. Let's use sphericals to parametrize; $\mathbf{S}' : 0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$, $\rho = 2$.

$$\mathbf{r}(\theta, \varphi) = \langle 2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi \rangle$$

$$\mathbf{r}_\theta = \langle -2 \sin \theta \sin \varphi, 2 \cos \theta \sin \varphi, 0 \rangle$$

$$\mathbf{r}_\varphi = \langle 2 \cos \theta \cos \varphi, 2 \sin \theta \cos \varphi, -2 \sin \varphi \rangle$$

$$\mathbf{r}_\theta \times \mathbf{r}_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 \sin \theta \sin \varphi & 2 \cos \theta \sin \varphi & 0 \\ 2 \cos \theta \cos \varphi & 2 \sin \theta \cos \varphi & -2 \sin \varphi \end{vmatrix}$$

$$= 4 \langle -\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\underline{\sin^2 \theta \sin \varphi \cos \varphi} - \underline{\cos^2 \theta \sin \varphi \cos \varphi} \rangle$$

And now the fun part,

$$|\mathbf{r}_\theta \times \mathbf{r}_\varphi| = 4 \sqrt{\cos^2 \theta \sin^4 \varphi + \sin^2 \theta \sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi}$$

$$= 4 \sqrt{\sin^2 \varphi [\sin^2 \varphi (\cos^2 \theta + \sin^2 \theta) + \cos^2 \varphi]}$$

$$= 4 \sqrt{\sin^2 \varphi}$$

$$= 4 \sin \varphi \quad (\text{notice } 0 \leq \varphi \leq \frac{\pi}{2} \Rightarrow \sin \varphi \geq 0.)$$

§13.6 #15 continued

One last algebraic question to resolve

$$\begin{aligned} x^2 z + y^2 z &= (2 \cos \theta \sin \varphi)^2 (2 \cos \varphi) + (2 \sin \theta \sin \varphi)^2 (2 \cos \varphi) \\ &= 8 \cos \varphi [\cos^2 \theta \sin^2 \varphi + \sin^2 \theta \sin^2 \varphi] \\ &= 8 \cos \varphi \sin^2 \varphi \end{aligned}$$

Thus,

$$\begin{aligned} \iint_S (x^2 z + y^2 z) dS &= \int_0^{\pi/2} \int_0^{2\pi} (8 \cos \varphi \sin^2 \varphi) \cdot (4 \sin \varphi) d\theta d\varphi \\ &= 64\pi \int_0^{\pi/2} \sin^3 \varphi \cos \varphi d\varphi \\ &= 64\pi \left(\frac{1}{4} \sin^4 \varphi \Big|_0^{\pi/2} \right) = [16\pi] \end{aligned}$$

§13.6 #24 Find $\iint_S \vec{F} \cdot d\vec{S}$ for S the hemisphere $x^2 + y^2 + z^2 = 25$, $z \geq 0$
with the orientation in the positive z -direction and $\vec{F} = \langle xz, x, y \rangle$.

The unit normal to the sphere is $\hat{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle = \frac{1}{5} \langle x, y, z \rangle$

$$\vec{F} \cdot \hat{r} = \frac{1}{5} \langle xz, x, y \rangle \cdot \langle x, y, z \rangle = \frac{1}{5} (x^2 z + xy + yz)$$

Then notice S is parametrized by $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq \pi$, and $\rho = 5$.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_S (\vec{F} \cdot \hat{r}) dS = \int_0^{\pi} \int_0^{\pi} \frac{1}{5} (x^2 z + xy + yz) \cdot \underline{5 \sin \varphi} d\theta d\varphi \\ &= \int_0^{\pi} \int_0^{\pi} 5 (125 \sin^2 \varphi \cos^2 \theta \cos \varphi + 25 \sin^2 \varphi \cos \theta \sin \theta + 25 \sin \theta \sin \varphi \cos \varphi) \sin \varphi d\theta d\varphi \quad \text{follows from # 15 with } \rho = 5 \\ &= \int_0^{\pi} \int_0^{\pi} 125 (5 \sin^3 \varphi \cos \varphi \cos^2 \theta + \sin^3 \varphi \cos \theta \sin \theta + \sin^2 \varphi \cos \varphi \sin \theta) d\theta d\varphi \\ &= 125 \left(\underbrace{\int_0^{\pi} \cos \theta d\theta}_{\text{zero}} \int_0^{\pi} \sin^3 \varphi \cos \varphi d\varphi + \int_0^{\pi} (-\cos^2 \varphi) \sin \varphi d\varphi \underbrace{\int_0^{\pi} \sin(\varphi) d\varphi}_{\text{zero}} + \int_0^{\pi} \sin^2 \varphi \cos \varphi d\varphi \int_0^{\pi} \sin \theta d\theta \right) \\ &= 125 \left(\frac{1}{3} \sin^3 \varphi \Big|_0^{\pi} \right) (-\cos \theta \Big|_0^{\pi}) \\ &= 0. \end{aligned}$$

§13.6 #38) Seawater has a density $\bar{\rho} = 1025 \text{ kg/m}^3$ and flows with a velocity field of $\mathbf{v} = \langle y, x, 0 \rangle$ in meters per second. Find flow-rate of seawater through $x^2 + y^2 + z^2 = 9$, $z \geq 0$ outward. (S') Using sphericals we have $\rho = 3$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi/2$. The surface integral of $\mathbf{v}\bar{\rho}$ over S' gives the flow rate through S' .

$$[\mathbf{v}\bar{\rho}] = \left(\frac{\text{m}}{\text{s}}\right)\left(\frac{\text{kg}}{\text{m}^3}\right) = \frac{\text{kg}}{\text{m}^2 \cdot \text{s}} \quad \text{integrate over area} \Rightarrow \frac{\text{kg}}{\text{s}} \quad (\text{a flow rate})$$

The parametrization of S is

$$\mathbf{r}(\theta, \varphi) = \langle 3\cos\theta\sin\varphi, 3\sin\theta\sin\varphi, 3\cos\varphi \rangle$$

$$\mathbf{r}_\theta = \langle -3\sin\theta\sin\varphi, 3\cos\theta\sin\varphi, 0 \rangle$$

$$\mathbf{r}_\varphi = \langle 3\cos\theta\cos\varphi, 3\sin\theta\cos\varphi, -3\sin\varphi \rangle$$

$$\mathbf{r}_\theta \times \mathbf{r}_\varphi = \langle -9\cos\theta\sin^2\varphi, -9\sin\theta\sin^2\varphi, -9\sin\varphi\cos\varphi \rangle$$

Now we use $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_\varphi) dA$ where $\mathbf{r}: D \rightarrow S$

And calculate for $\mathbf{F} = \rho\mathbf{v}$,

$$D = [0, 2\pi] \times [0, \pi/2].$$

$$\begin{aligned} \iint_S \rho\mathbf{v} \cdot d\mathbf{S} &= 1025 \int_0^{2\pi} \int_0^{\pi/2} \langle y, x, 0 \rangle \cdot \langle \cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, \sin\varphi\cos\varphi \rangle (-9) d\theta d\varphi \\ &= -9(1025) \int_0^{2\pi} \int_0^{\pi/2} [y\cos\theta\sin^2\varphi + x\sin\theta\sin^2\varphi] d\theta d\varphi \\ &= -9(1025) \int_0^{2\pi} \int_0^{\pi/2} [3\sin\theta\cos\theta\sin^3\varphi + 3\sin\theta\cos\theta\sin^3\varphi] d\theta d\varphi \\ &= -54(1025) \int_0^{2\pi} \sin^3\varphi d\varphi \int_0^{\pi/2} \sin\theta\cos\theta d\theta \\ &= -54(1025) \left(\int_0^{2\pi} (1 - \cos^2\varphi) \sin\varphi d\varphi \right) \left(\frac{\sin^2\theta}{2} \Big|_0^{\pi/2} \right) \\ &= 0. \end{aligned}$$

Remark: this sol^t is flawed in that we used the inward-pointing normal. Instead we should have used $\mathbf{r}_\varphi \times \mathbf{r}_\theta$ which points out (take $\theta = \varphi = \pi/4$ for example, you'd get $\langle \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \rangle$).

§13.6 #38 Alternate sol^t. The eq $x^2 + y^2 + z^2 = 9$ is a level surface of $G(x, y, z) = x^2 + y^2 + z^2$, note $\nabla G = 2\langle x, y, z \rangle$. A unit normal is simply $\hat{r} = \frac{1}{3}\langle x, y, z \rangle$. We can see it points outward and $\hat{r} \cdot \hat{r} = \frac{1}{9}(x^2 + y^2 + z^2) = \frac{1}{9}(9) = 1$ (it's a unit vect.)

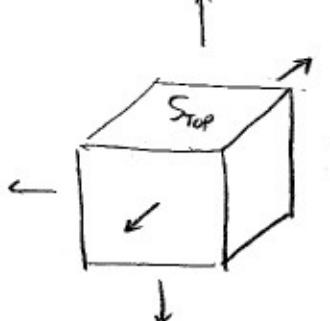
$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \bar{\rho}_3 \iint_D \langle y, x, 0 \rangle \cdot \langle x, y, z \rangle dx dy \quad D = \{(x, y) \mid x^2 + y^2 \leq 9\} \\ &= \bar{\rho}_3 \iint_D 2xy dx dy \\ &= 2\bar{\rho}_3 \int_0^3 \int_0^{2\pi} 27 \cos \theta \sin \theta dr d\theta \quad \begin{array}{l} x = 3 \cos \theta \\ y = 3 \sin \theta \\ dx dy = 3 dr d\theta \end{array} \\ &= 0. \end{aligned}$$

§13.6 #40 Gauss' Law says $Q = \epsilon_0 \iint_S \vec{E} \cdot d\vec{S}$.

Suppose $\vec{E} = \langle x, y, z \rangle$ and $S = [1, 1] \times [-1, 1] \times [-1, 1]$

By symmetry the flux through each of the six faces of S will be the same. Calculate the top face $z = 1, -1 \leq x, y \leq 1$ which has unit normal \hat{k}

$$\begin{aligned} \iint_{S_{\text{top}}} \vec{E} \cdot d\vec{S} &= \int_{-1}^1 \int_{-1}^1 (\langle x, y, 1 \rangle \cdot \hat{k}) dx dy \\ &= \int_{-1}^1 \int_{-1}^1 dx dy \\ &= (2)(2) = 4 \quad \therefore \quad \iint_S \vec{E} \cdot d\vec{S} = 24 \end{aligned}$$



this \vec{E} field
is purely radial
(no θ, ϕ dependence)

$$\therefore Q = 24 \epsilon_0$$