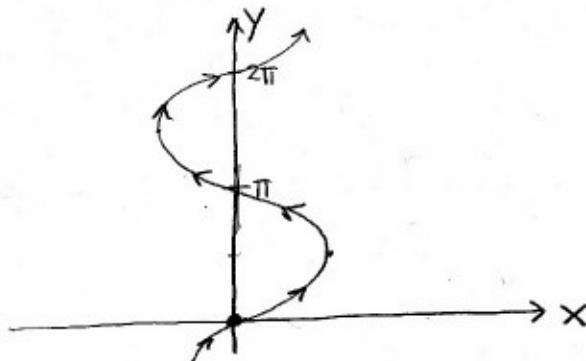


§10.1 #5 Sketch $r(t) = \langle \sin t, t \rangle$.

t	x	y
0	0	0
$\pi/4$	$1/\sqrt{2}$	$\pi/4$
$\pi/2$	1	$\pi/2$
$3\pi/4$	$-1/\sqrt{2}$	$3\pi/4$
π	0	π
$5\pi/4$	$1/\sqrt{2}$	$5\pi/4$

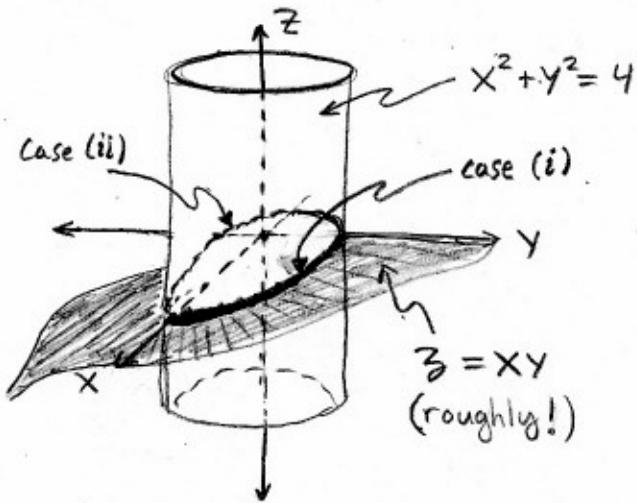


§10.1 #25 At what point does $r(t) = \langle t, 0, 2t - t^2 \rangle$ intersect the paraboloid $z = x^2 + y^2$? As usual we find intersection point by assuming both eq's hold,

$$\begin{aligned} z = x^2 + y^2 &\Rightarrow 2t - t^2 = t^2 + 0^2 \\ &\Rightarrow 2t - 2t^2 = 2t(1-t) = 0 \\ &\Rightarrow t = 0 \text{ or } t = 1 \end{aligned}$$

the points of intersection are $(0, 0, 0)$ and $(1, 0, 1)$.

§10.1 #30 Find vector function which represents the intersection of the surfaces $x^2 + y^2 = 4$ and $z = xy$. Lets use x as the parameter. Then



$$\begin{aligned} y &= \pm \sqrt{4 - x^2} \\ z &= x(\pm \sqrt{4 - x^2}) \end{aligned}$$

the question then is (+) or (-) when and where? I'll break it up into cases.

- (i) $y \geq 0 \Rightarrow r(x) = \langle x, \sqrt{4-x^2}, x\sqrt{4-x^2} \rangle, -2 \leq x \leq 2$.
- (ii) $y \leq 0 \Rightarrow r(x) = \langle x, -\sqrt{4-x^2}, -x\sqrt{4-x^2} \rangle, -2 \leq x \leq 2$.

I suppose we could paste these together by shifting the parameter on either (i) or (ii). There are other ways, for example,

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 2\sin(2t), \quad 0 \leq t \leq 2\pi$$

§10.1 #33 find curve of intersection of $z = \sqrt{x^2 + y^2}$
 and the plane $z = 1 + y$. Again use x as parameter,
 note $y = z - 1 \Rightarrow z = \sqrt{x^2 + (z-1)^2}$
 $\Rightarrow z^2 = x^2 + (z-1)^2$
 $\Rightarrow z^2 = x^2 + z^2 - 2z + 1$
 $\Rightarrow 2z = x^2 + 1$
 $\Rightarrow z = \frac{1}{2}(x^2 + 1)$
 $y = z - 1 = \frac{1}{2}x^2 + \frac{1}{2} - 1 = \frac{1}{2}(x^2 - 1) = y$

So if you wish we may introduce t as a parameter then

$$\boxed{r(t) = \langle t, \frac{1}{2}(t^2 - 1), \frac{1}{2}(t^2 + 1) \rangle} \quad (\text{there are other answers.})$$

§10.1 #37 Consider the following trajectories, do they collide? For $t \geq 0$

$$r_1(t) = \langle t^2, 7t-12, t^2 \rangle \text{ and } r_2(t) = \langle 4t-3, t^2, 5t-6 \rangle$$

For vector functions to be equal we need each component to match with the corresponding component. That is,

$$x_1 = x_2 \Rightarrow t^2 = 4t-3 \Rightarrow t^2 - 4t + 3 = (t-1)(t-3) = 0 \therefore \boxed{t=1 \text{ or } t=3}$$

$$y_1 = y_2 \Rightarrow 7t-12 = t^2 \Rightarrow t^2 - 7t + 12 = (t-3)(t-4) = 0 \therefore \boxed{t=3 \text{ or } t=4}$$

$$z_1 = z_2 \Rightarrow t^2 = 5t-6 \Rightarrow t^2 - 5t + 6 = (t-3)(t-2) = 0 \therefore \boxed{t=3 \text{ or } t=2}$$

We find $x_1 = x_2$ at $t=1$, $y_1 = y_2$ at $t=4$ and $z_1 = z_2$ at $t=2$.

But only at $t=3$ do we get $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$, this means the particles will collide at $t=3$

§10.1 #38 the question of collision is $r_1(t) \stackrel{?}{=} r_2(t)$ for some $t \geq 0$?

I leave that to you. The question of intersection is a bit different we should study $r_1(t) \stackrel{?}{=} r_2(s)$, can we find $s, t \geq 0$ so that the positions match-up (for possibly different times)

$$\left. \begin{array}{l} x: t = 1+2s \\ y: t^2 = 1+6s \\ z: t^3 = 1+14s \end{array} \right\} \underline{1 = t-2s = t^2-6s = t^3-14s} \quad \therefore \text{ note } s=0 \text{ works if we make } t=1.$$

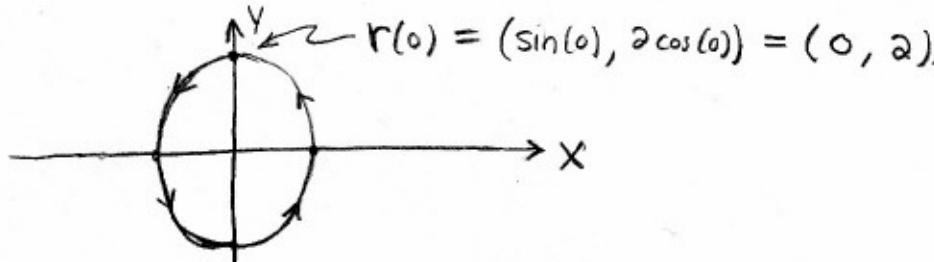
So yes the paths intersect at $r_1(1) = r_2(0) = \langle 1, 1, 1 \rangle$

there is one other place they intersect, can you find it?

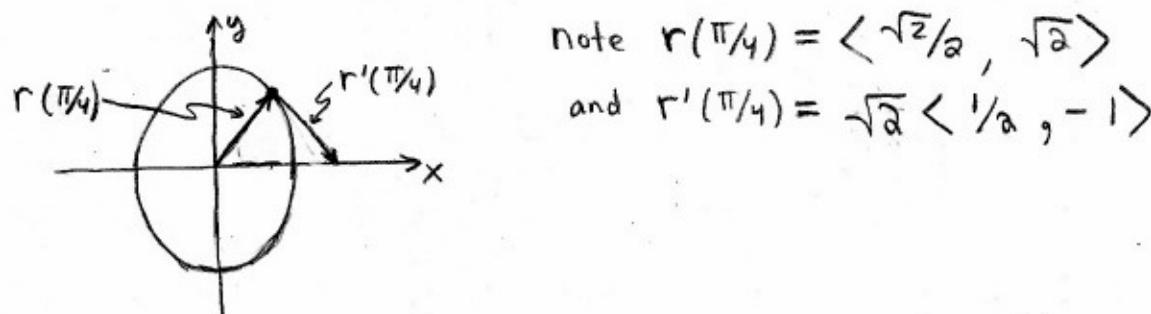
§10.2#5 Sketch curve $r(t)$, find $r'(t)$ and plot $r'(\pi/4)$. Given

$$r(t) = \langle \sin t, 2\cos t \rangle$$

Notice $\sin^2 t + \cos^2 t = \sin^2 t + \frac{(2\cos t)^2}{2^2} = x^2 + \frac{y^2}{2^2} = 1$
this is an ellipse.



Next $r'(t) = \langle \cos t, -2\sin t \rangle$ so $r'(\pi/4) = \langle \cos(\pi/4), -2\sin(\pi/4) \rangle$



§10.2#10 Let $r(t) = \langle \cos 3t, t, \sin 3t \rangle$ differentiate,

$$r'(t) = \langle -3\sin 3t, 1, 3\cos 3t \rangle$$

§10.2#13 Let $r(t) = a + tb + t^2c$ where a, b, c are constant vectors. Then using Th²(3) part 3.

$$\begin{aligned} r'(t) &= \frac{d}{dt} [a + tb + t^2c] \\ &= \frac{da}{dt} + \frac{dt}{dt}b + t \frac{db}{dt} + \frac{d}{dt}(t^2)c + t^2 \frac{dc}{dt} \\ &= [b + 2tc] \quad \text{since } \frac{da}{dt} = \frac{db}{dt} = \frac{dc}{dt} = 0 \end{aligned}$$

because a, b, c are constant vectors.

§10.2#16 $r(t) = \langle 2\sin t, 2\cos t, \tan t \rangle$ • we calculate the unit tangent vector at $r(\pi/4)$.

$$r'(t) = \langle 2\cos t, -2\sin t, \sec^2 t \rangle$$

$$T(\pi/4) = \frac{r'(\pi/4)}{\|r'(\pi/4)\|} = \frac{1}{\|\langle -\sqrt{2}, -\sqrt{2}, 2 \rangle\|} \langle -\sqrt{2}, -\sqrt{2}, 2 \rangle \quad \sec^2(\pi/4) = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2.$$

$$= \frac{1}{\sqrt{2+2+4}} \langle -\sqrt{2}, -\sqrt{2}, 2 \rangle = \frac{1}{2\sqrt{2}} \langle -\sqrt{2}, -\sqrt{2}, 2 \rangle$$

$$\therefore T(\pi/4) = \langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2\sqrt{2}} \rangle$$

§10.2 #20 Find parametric eq's to tangent line to curve at the given point; $(-1, 1, 1)$ for

$$\mathbf{r}(t) = \langle t^2 - 1, t^2 + 1, t + 1 \rangle$$

$$\mathbf{r}'(t) = \langle 2t, 2t, 1 \rangle$$

What t -value gives $\mathbf{r}(t) = (-1, 1, 1)$? Well we'd need $t^2 - 1 = -1$, $t^2 + 1 = 1$ and $t + 1 = 1$, clearly $t=0$ is the correct value.

Note $\mathbf{r}'(0) = \langle 0, 0, 1 \rangle$ thus the tangent line is

$$\mathbf{l}(s) = (-1, 1, 1) + s\langle 0, 0, 1 \rangle$$

or if it makes you happy

$$x = -1, y = 1, z = 1 + s$$

I introduce s to reduce possible confusion with t which is used already

§10.2 #27 Consider $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ these intersect at $t=0$ where $\mathbf{r}_1(0) = \mathbf{r}_2(0) = \langle 0, 0, 0 \rangle$. The angle of intersection is the angle between their tangents.

$$\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle \Rightarrow \mathbf{r}_1'(0) = \langle 1, 0, 0 \rangle$$

$$\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle \Rightarrow \mathbf{r}_2'(0) = \langle 1, 2, 1 \rangle$$

Find angle via dot-product. Note that $\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0) = 1$ while $|\mathbf{r}_1'(0)| = 1$ and $|\mathbf{r}_2'(0)| = \sqrt{1+4+1} = \sqrt{6}$ thus

$$1 = \sqrt{6} \cos \theta \Rightarrow \cos \theta = \frac{1}{\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 66^\circ$$

§10.2 #30 Recall that $\int \frac{dt}{1+t^2} = \tan^{-1}(t) + C$ & $\int \frac{2t dt}{1+t^2} = \int \frac{d(1+t^2)}{1+t^2} = \ln|1+t^2| + C$ thus we calculate,

$$\begin{aligned} \int_0^1 \left\langle 0, \frac{4}{1+t^2}, \frac{2t}{1+t^2} \right\rangle dt &= \left\langle \int_0^1 0 \cdot dt, \int_0^1 \frac{4 dt}{1+t^2}, \int_0^1 \frac{2t dt}{1+t^2} \right\rangle \\ &= \left\langle 0, 4\tan^{-1}(t) \Big|_0^1, \ln|1+t^2| \Big|_0^1 \right\rangle \\ &= \left\langle 0, 4\tan^{-1}(1), \ln(2) \right\rangle \\ &= \boxed{\langle 0, \pi, \ln(2) \rangle} \end{aligned}$$

Notice, $\tan(\pi/4) = 1 \therefore \tan^{-1}(1) = \pi/4$.

§10.2 #43 Suppose r is a vector function such that r'' exists.

$$\begin{aligned}\frac{d}{dt} [r(t) \times r'(t)] &= r'(t) \times r'(t) + r(t) \times r''(t) \quad : \text{using Thm } (3), \\ &= r(t) \times r''(t) \quad : \underline{r'(t) \times r'(t) = 0}.\end{aligned}$$

There it be $\boxed{\frac{d}{dt} [r(t) \times r'(t)] = r(t) \times r''(t)}$.

§10.2 #45 Suppose $r(t) \neq 0$. Note $|r(t)|^2 = r(t) \cdot r(t)$. Then

$$\frac{d}{dt} |r(t)|^2 = \frac{d}{dt} [r(t) \cdot r(t)] = r'(t) \cdot r(t) + r(t) \cdot r'(t) = 2r(t) \cdot r'(t).$$

Yet on the other hand we can calculate via chain-rule,

$$\frac{d}{dt} |r(t)|^2 = 2|r(t)| \frac{d}{dt} [|r(t)|]$$

So we have that

$$2|r(t)| \frac{d}{dt} [|r(t)|] = 2r(t) \cdot r'(t)$$

$$\Rightarrow \boxed{\frac{d}{dt} |r(t)| = \frac{1}{|r(t)|} r(t) \cdot r'(t)}$$

we can divide by $|r(t)|$ since it is non zero as $r(t) \neq 0$.

I think it's more fun to write $\vec{r} = r\hat{r}$ so $\frac{dr}{dt} = \hat{r} \cdot r'$.

§10.3 #2 Find length of curve below, for $0 \leq t \leq \pi$,

$$r(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$$

$$r'(t) = \langle 2t, \cos t - \cos t + t \sin t, -\sin t + \sin t + t \cos t \rangle$$

$$r'(t) = \langle 2t, t \sin t, t \cos t \rangle$$

$$r'(t) \cdot r'(t) = 4t^2 + t^2(\sin^2 t + \cos^2 t) = 5t^2. \therefore \boxed{|r'(t)|^2 = 5t^2}.$$

$$s = \int_0^\pi |r'(t)| dt = \int_0^\pi \sqrt{5t^2} dt = \sqrt{5} \frac{t^2}{2} \Big|_0^\pi = \boxed{\frac{\pi^2 \sqrt{5}}{2}}$$

Remark: most arclength integrals are not this nice. Usually the $\sqrt{\ }$ forces us to resort to numerics.

§10.3 #11 Let $r(t) = \langle 2\sin t, 5t, 2\cos t \rangle$ find $T(t)$, $N(t)$ and the curvature.

$$r'(t) = \langle 2\cos t, 5, -2\sin t \rangle$$

$$r'(t) \cdot r'(t) = 4\cos^2 t + 25 + 4\sin^2 t = 29. \therefore |r'(t)| = \sqrt{29}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \boxed{\frac{1}{\sqrt{29}} \langle 2\cos t, 5, -2\sin t \rangle} = T(t) \quad \text{Unit Tangent}$$

$$T'(t) = \frac{1}{\sqrt{29}} \langle -2\sin t, 0, -2\cos t \rangle$$

$$T'(t) \cdot T'(t) = \frac{1}{29} (4\sin^2 t + 4\cos^2 t) = \frac{4}{29} \therefore |T'(t)| = \frac{2}{\sqrt{29}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\sqrt{29}}{2} \frac{1}{\sqrt{29}} \langle -2\sin t, 0, -2\cos t \rangle$$

$$\therefore N(t) = \langle -\sin t, 0, -\cos t \rangle \quad \text{Unit Normal.}$$

$$K(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{2}{\sqrt{29}} \frac{1}{\sqrt{29}} = \boxed{\frac{2}{29}} = K(t) \quad \text{curvature.}$$

§10.3 #18 $r(t) = \langle e^t \cos t, e^t \sin t, t \rangle$ note $r(0) = \langle 1, 0, 0 \rangle$.

$$r'(t) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), 1 \rangle \quad \text{note } r'(0) = \langle 1, 1, 1 \rangle.$$

$$r''(t) = \langle e^t(-2\sin t), e^t(2\cos t), 0 \rangle \quad \text{note } r''(0) = \langle 0, 2, 0 \rangle.$$

$$r'(0) \cdot r'(0) = \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 3 \therefore |r'(0)| = \sqrt{3}$$

$$r'(0) \times r''(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2\hat{i} + 2\hat{k} = \langle -2, 0, 2 \rangle$$

$$|r'(0) \times r''(0)| = |\langle -2, 0, 2 \rangle| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$K(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{2\sqrt{2}}{(\sqrt{3})^3} = \frac{2\sqrt{2}}{3\sqrt{3}} = \boxed{\frac{2\sqrt{6}}{9}} = K(0)$$

§10.3 #39 Find the eq's of the normal plane and osculating plane of the curve $r(t) = \langle 2\sin(3t), t, 2\cos(3t) \rangle$ at $(0, \pi, -2)$.

$$r'(t) = \langle 6\cos(3t), 1, -6\sin(3t) \rangle \Rightarrow |r'(t)| = \sqrt{37}$$

$$r''(t) = \langle -18\sin(3t), 0, -18\cos(3t) \rangle \Rightarrow |r''(t)| = 18$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{37}} \langle 6\cos(3t), 1, -6\sin(3t) \rangle$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\cancel{\frac{1}{\sqrt{37}}}}{\cancel{\frac{1}{\sqrt{37}}}} \frac{r''(t)}{|r''(t)|} = \frac{1}{18} \langle -18\sin 3t, 0, -18\cos 3t \rangle$$

We could continue calculating for arbitrary t but $t=\pi$ gives the point $(0, \pi, -2)$. Notice

$$T(\pi) = \frac{1}{\sqrt{37}} \langle -6, 1, 0 \rangle$$

$$N(\pi) = \frac{1}{18} \langle 0, 0, 18 \rangle = \hat{k}$$

$$B(\pi) \equiv T(\pi) \times N(\pi) = \frac{1}{\sqrt{37}} (-6\hat{i} + \hat{j}) \times \hat{k} = \frac{-6}{\sqrt{37}} \hat{j} + \frac{1}{\sqrt{37}} \hat{i}$$

$$B(\pi) = \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$$

Plane determined by N and B is the "Normal Plane", it has normal $N \times B = T$ so we can use $\langle -6, 1, 0 \rangle$ and the point $(0, \pi, -2)$ to give

$$\boxed{-6x + y - \pi = 0 \text{ Normal Plane}}$$

The osculating plane has normal along $B = \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$ dropping the ugly $1/\sqrt{37}$ still gives same direction and,

$$\boxed{x + 6(y - \pi) = 0 \text{ Osculating Plane}}$$

§ 10.3 #45 Show that $\frac{dT}{ds} = \kappa N$ where T , κ and N are understood to be functions of arclength s . By definition

$$\kappa = \left| \frac{dT}{ds} \right|$$

Also by definition $N(t) = \frac{T'(t)}{\|T'(t)\|}$. Likewise,

$$N(s) = \frac{\frac{dT}{ds}}{\left| \frac{dT}{ds} \right|} = \frac{dT}{\left| \frac{dT}{ds} \right|} \therefore \boxed{\frac{dT}{ds} = \kappa N}$$

Notice we abuse notation here since technically $N(t)$ and $N(s)$ are different funcs, that aside. We could find this formula starting with $N(t)$. Since $\frac{dT}{dt} = \frac{dT}{ds} \frac{ds}{dt}$ means that in prime notation $T'(t) = T'(s) \frac{ds}{dt}$.

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{T'(s) \frac{ds}{dt}}{\|T'(s) \frac{ds}{dt}\|} = \frac{\frac{ds}{dt}}{\left| \frac{ds}{dt} \right|} \frac{T'(s)}{\|T'(s)\|} = \frac{T'(s)}{\|T'(s)\|}$$

Since $\frac{ds}{dt} > 0$ which can be shown as follows,

$$\frac{d}{dt} \int_a^t \|r'(u)\| du \stackrel{(F.T.C.)}{=} \|r'(t)\| > 0$$

What I've shown here is that $N(t)$ and $N(t(s))$ are defined the same way. That is our definitions are parametrization independent.

$$\left(\begin{array}{l} F.T.C = \text{Fundamental Thm of Calculus, In it's simplest form,} \\ \frac{d}{dx} \int_a^x f(t) dt = f(x) \end{array} \right)$$

§ 10.3 #47 Recall $T = \frac{r'}{|r'|}$, $N = \frac{T'}{|T'|}$ and $B = T \times N$, moreover by construction $T \cdot T = N \cdot N = B \cdot B = 1$.

(a.) $B \cdot B = 1$

$$\frac{d}{ds}(B \cdot B) = B' \cdot B + B \cdot B' = 2B \cdot B' = \frac{d}{ds}(1) = 0.$$

Thus $\boxed{\frac{dB}{ds} \cdot B = 0}$ they're perpendicular.

(b.) $\frac{dB}{ds} = \frac{d}{ds}[T \times N] = \frac{dT}{ds} \times N + T \times \frac{dN}{ds}$
 $= \cancel{N \times N} + T \times \frac{dN}{ds}$ (using #45)

Thus $\frac{dB}{ds} = T \times \frac{dN}{ds}$ so clearly $\boxed{\frac{dB}{ds} \cdot T = 0}$

since $(A \times B) \cdot A = 0$, cross product's perpendicular to both its inputs.

(c.) Any vector can be written as a linear combination of T, N, B thus,

$$\frac{dB}{ds} = aT + bN + cB$$

Notice that $\frac{dB}{ds} \cdot T = a$ and $\frac{dB}{ds} \cdot B = c$ since we know $N \cdot T = B \cdot T = N \cdot B = 0$. This leaves

$$\frac{dB}{ds} = bN \equiv \boxed{-T(s)N = \frac{dB}{ds}}$$

this formula defines the torsion T of the curve.

(d.) A plane curve has $y = f(x)$ so $r(x) = \langle x, f(x), 0 \rangle$

$$\text{hence } r'(x) = \langle 1, f'(x), 0 \rangle \therefore |r'(x)| = \sqrt{1 + [f'(x)]^2}$$

$$T(x) = \frac{1}{\sqrt{1 + [f'(x)]^2}} \langle 1, f'(x), 0 \rangle$$

Continued \rightarrow

§10.3 #47d Show $y = f(x)$ has $\tau = 0$.

$$T(x) = \left\langle \frac{1}{\sqrt{1+[f'(x)]^2}}, \frac{f'(x)}{\sqrt{1+[f'(x)]^2}}, 0 \right\rangle = \langle \alpha, \beta, 0 \rangle$$

$$\begin{aligned} T'(x) &= \left\langle \frac{f''(x)f''(x)}{(1+(f'(x))^2)^{3/2}}, \frac{f''(x)\sqrt{1+[f'(x)]^2} - \frac{f'(x)f''(x)}{\sqrt{1+(f'(x))^2}}f'(x)}{1+[f'(x)]^2}, 0 \right\rangle \\ &= \frac{f''}{(1+(f')^2)^{3/2}} \langle f', 1 + (f')^2 - (f')^2, 0 \rangle \\ &= \frac{f''}{(1+(f')^2)^{3/2}} \langle f', 1, 0 \rangle \end{aligned}$$

$$N(x) = \frac{T'(x)}{|T'(x)|} = \frac{1/(1+(f')^2)^{3/2}/\sqrt{1+(f')^2}}{|f''|} \frac{f''}{(1+(f')^2)^{3/2}} \langle f', 1, 0 \rangle$$

$$N(x) = \frac{f''}{|f''|} \sqrt{1+(f')^2} \langle f', 1, 0 \rangle = \langle \gamma, \delta, 0 \rangle$$

$$B(x) = \begin{vmatrix} i & j & k \\ \alpha & \beta & 0 \\ \gamma & \delta & 0 \end{vmatrix} = \hat{k}(\alpha\delta - \beta\gamma) = \langle 0, 0, \alpha\delta - \beta\gamma \rangle$$

Consider that $\frac{dB}{ds} = -\tau(s)N \Rightarrow \frac{dx}{ds} \frac{dB}{dx} = -\tau N$

But $\frac{dB}{dx} = \langle 0, 0, \frac{d}{dx}[\alpha\delta - \beta\gamma] \rangle$ whereas $N = \langle \gamma, \delta, 0 \rangle$

thus $\frac{dB}{dx}$ is perpendicular to $N \therefore \boxed{\tau = 0}$

Remark: We could have been much more vague about the derivatives involved, much work above was pointless.

§10.3 #48 Prove $\frac{dN}{ds} = -\kappa T + \tau B$, note $N = B \times T$

$$\begin{aligned}\frac{dN}{ds} &= \frac{d}{ds}(B \times T) = \frac{dB}{ds} \times T + B \times \frac{dT}{ds} \\ &= -T N \times T + B \times (\kappa N) \\ &= \boxed{\tau B - \kappa T = \frac{dN}{ds}}\end{aligned}$$

Since just as $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$ we also have $T \times N = B$, $N \times B = T$ and $B \times T = N$.

§10.4 #9 Let the position be $r(t) = \langle t^2+1, t^3, t^2-1 \rangle$ assume $t \geq 0$,

$$\text{velocity } = r'(t) = \boxed{v(t) = \langle 2t, 3t^2, 2t \rangle}$$

$$\text{acceleration } = r''(t) = \boxed{a(t) = \langle 2, 6t, 2 \rangle}$$

$$\text{Speed } = |r'(t)| = |v(t)| = \sqrt{4t^2 + 9t^4 + 4t^2} = \boxed{t\sqrt{8 + 9t^2}}$$

§10.4 #14 Find $v = \frac{dr}{dt}$ and $r(t)$ given $a(t) = \langle 2, 6t, 12t^2 \rangle$

and the initial conditions $v(0) = \langle 1, 0, 0 \rangle$ and $r(0) = \langle 0, 1, -1 \rangle$

$$a = \frac{dv}{dt} \Rightarrow \int_{v(0)}^{v(t)} dv = \int_0^t a dt \Rightarrow v(t) - v(0) = \int_0^t \langle 2, 6t, 12t^2 \rangle dt$$

$$\text{Thus } v(t) = \langle 1, 0, 0 \rangle + \langle 2t, 3t^2, 4t^3 \rangle = \boxed{\langle 2t+1, 3t^2, 4t^3 \rangle} = v$$

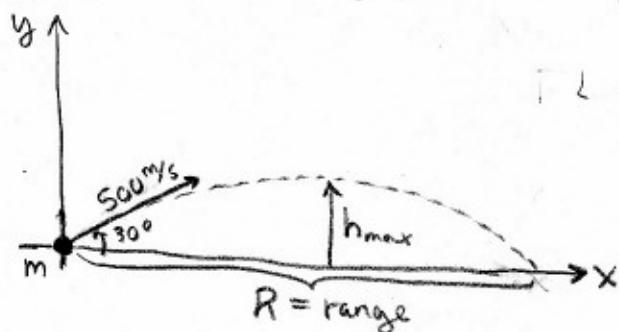
$$v = \frac{dr}{dt} \Rightarrow \int_{r(0)}^{r(t)} dr = \int_0^t v dt = \int_0^t \langle 2\tilde{t}+1, 3\tilde{t}^2, 4\tilde{t}^3 \rangle d\tilde{t}$$

$$\Rightarrow r(t) = \langle 0, 1, -1 \rangle + \langle t^2+t, t^3, t^4 \rangle$$

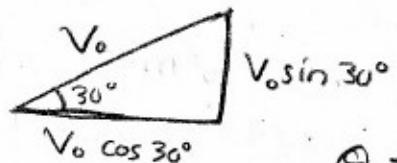
$$\boxed{r(t) = \langle t^2+t, t^3+1, t^4-1 \rangle}$$

- We can easily check that our answers match the given data, $r(0) = \langle 0, 1, -1 \rangle$ and $v(0) = \langle 1, 0, 0 \rangle$ its good to check answers.

§10.4 #21 Find range, max height, speed at impact. $g = 9.8 \text{ m/s}^2$.



$$V_0 = \langle 500 \cos 30^\circ, 500 \sin 30^\circ \rangle$$



$$\theta = 30^\circ$$

Newton's 2nd law says that $F = ma$ meaning,

$$m \langle 0, -g \rangle = m \langle x'', y'' \rangle$$

m cancels, ($m \neq 0$ we assume). Integrate both sides

$$\int_0^t \langle 0, -g \rangle dt = \int_0^t \langle x'', y'' \rangle dt$$

$$\Rightarrow \langle 0, -gt \rangle = \left\langle \int_0^t \frac{d}{dt} \left(\frac{dx}{dt} \right) dt, \int_0^t \frac{d}{dt} \left(\frac{dy}{dt} \right) dt \right\rangle$$

$$= \left\langle \frac{dx}{dt} - \frac{dx}{dt} \Big|_{t=0}, \frac{dy}{dt} - \frac{dy}{dt} \Big|_{t=0} \right\rangle$$

$$= \langle V_x(t) - V_0 \cos \theta, V_y(t) - V_0 \sin \theta \rangle$$

Thus $V(t) = \langle V_0 \cos \theta, V_0 \sin \theta - gt \rangle = \frac{dr}{dt}$

$$\int_{r(0)}^{r(t)} dr = r(t) - r(0) = \int_0^t \langle V_0 \cos \theta, V_0 \sin \theta - gt \rangle dt$$

$$\Rightarrow \underline{r(t)} = \langle V_0 \cos \theta t, V_0 \sin \theta t - \frac{1}{2} g t^2 \rangle$$

$$y=0 \text{ when } V_0 \sin \theta t - \frac{1}{2} g t^2 = t(V_0 \sin \theta - \frac{1}{2} g t) = 0$$

which gives 2 sol's namely $t_1 = 0$ and $t_2 = 2V_0 \sin \theta / g$.

$$x(t_2) = V_0 \cos \theta \cdot 2V_0 \sin \theta / g = \boxed{\frac{V_0^2 \sin(2\theta)}{g}} = R$$

$$\text{Max height } \frac{dy}{dt} = 0 = V_0 \sin \theta - gt \therefore t_{\max} = \frac{V_0 \sin \theta}{g}$$

Note $y'' = -g < 0 \therefore$ is max by 2nd derivative test

$$\text{thus } h_{\max} = V_0 \sin \theta \cdot \frac{V_0 \sin \theta}{g} - \frac{g}{2} \left(\frac{V_0 \sin \theta}{g} \right)^2 = \boxed{\frac{V_0^2 \sin^2 \theta}{2g} = h_{\max}}$$

§10.4 #21 continued

(You may put #'s in if you wish.)

the time of impact was shown to be $t_2 = 2V_0 \sin \theta / g$

Now calculate the speed $|V(t)|$ at time $t=t_2$.

$$\begin{aligned} V(t_2) &= \langle V_0 \cos \theta, V_0 \sin \theta - g \cdot (2V_0 \sin \theta / g) \rangle \\ &= \langle V_0 \cos \theta, V_0 \sin \theta - 2V_0 \sin \theta \rangle \\ &= \langle V_0 \cos \theta, -V_0 \sin \theta \rangle \quad \leftarrow (\text{makes sense.}) \end{aligned}$$

$$\text{Thus } |V(t_2)| = \sqrt{V_0^2 \cos^2 \theta + V_0^2 \sin^2 \theta} = \sqrt{V_0^2} = V_0$$

Remark: many of you could do much of this without algebra and/or calculus. That will get marginal partial credit. The point here is to learn to use calculus to derive results. Use your intuition (if you have it for these kind of problems) as a check.

§10.4 #31 find tangential & normal components of $a(t)$ for

$$r(t) = \langle 3t - t^3, 3t^2, 0 \rangle$$

$$\begin{aligned} r'(t) &= \langle 3 - 3t^2, 6t, 0 \rangle \Rightarrow |r'(t)| = \sqrt{9 - 18t^2 + 9t^4 + 36t^2} \\ r''(t) &= \langle -6t, 6, 0 \rangle \\ &= \sqrt{9 + 9t^4 + 18t^2} \\ &= 3\sqrt{t^4 + 2t^2 + 1} \\ &= 3(t^2 + 1) = |r'(t)| \end{aligned}$$

Thus we find

$$T(t) = \frac{1}{3(t^2+1)} \langle 3(1-t^2), 6t, 0 \rangle$$

$$T(t) = \left\langle \frac{1-t^2}{1+t^2}, \frac{6t}{1+t^2}, 0 \right\rangle$$

$$T'(t) = \left\langle \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}, \frac{2(1+t^2) - 2t(2t)}{(1+t^2)^2}, 0 \right\rangle$$

$$T'(t) = \frac{1}{(1+t^2)^2} \langle -4t, 2-2t^2, 0 \rangle$$

$$|T'(t)| = \frac{1}{(1+t^2)^2} \sqrt{16t^2 + 4(1-2t^2 + t^4)} = \frac{2}{(1+t^2)^2} \sqrt{\underbrace{t^4 + 2t^2 + 1}_{(t^2+1)^2}}$$

$$|T'(t)| = 2 / (1+t^2)$$

§ 10.4 #31 We've calculated T and N as follows,

$$T = \frac{1}{1+t^2} \langle 1-t^2, 2t, 0 \rangle$$

$$N = \frac{T'}{|T'|} = \left(\frac{1+t^2}{2}\right) \cdot \frac{1}{(1+t^2)^2} \langle -4t, 2(1-t^2), 0 \rangle$$

$$N = \frac{1}{1+t^2} \langle -2t, 1-t^2, 0 \rangle$$

Now we may find a_T and a_N for $a = 6 \langle -t, 1, 0 \rangle$

$$a_T = a \cdot T = \frac{6}{1+t^2} \langle -t, 1, 0 \rangle \cdot \langle 1-t^2, 2t, 0 \rangle$$

$$a_T = \frac{6}{1+t^2} (t^3 - t + 2t) = \frac{6t(t^2+1)}{1+t^2} = \boxed{6t = a_T}$$

$$a_N = a \cdot N = \frac{6}{1+t^2} \langle -t, 1, 0 \rangle \cdot \langle -2t, 1-t^2, 0 \rangle$$

$$a_N = \frac{6}{1+t^2} (2t^2 + 1-t^2) = \frac{6(1+t^2)}{1+t^2} = \boxed{6 = a_N}$$

§ 10.4 #36 Angular Momentum $L(t) = mr(t) \times v(t)$ and then
Torque is $T(t) = mr(t) \times a(t)$.

$$\begin{aligned} \frac{dL}{dt} &= \frac{d}{dt} [mr(t) \times v(t)] \\ &= m \left[\frac{dr}{dt} \times v(t) + r(t) \times \frac{dv}{dt}(t) \right] \\ &= m \left[\frac{dr}{dt} \times \frac{dr}{dt} + r(t) \times a(t) \right] \\ &= mr(t) \times a(t) \\ &= \boxed{T(t) = \frac{dL}{dt}} \end{aligned}$$