

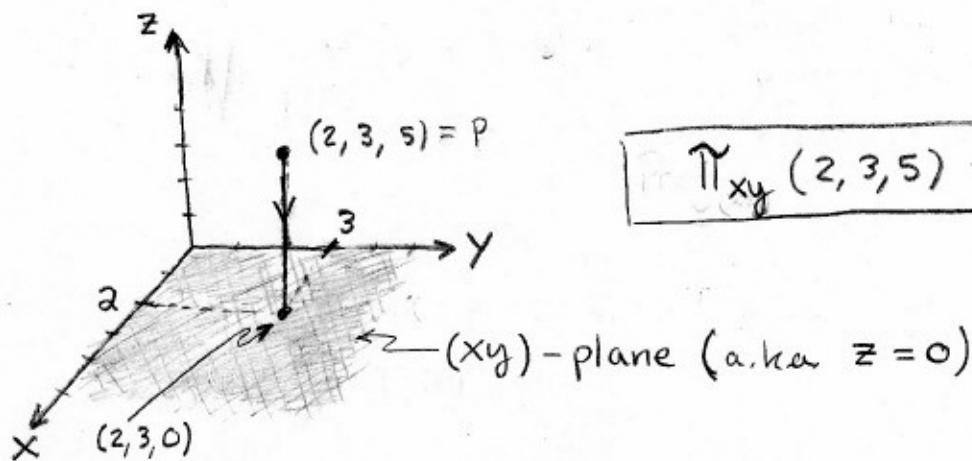
SOLUTIONS TO SELECT HOMEWORKS FOR CALCULUS III

(HI)

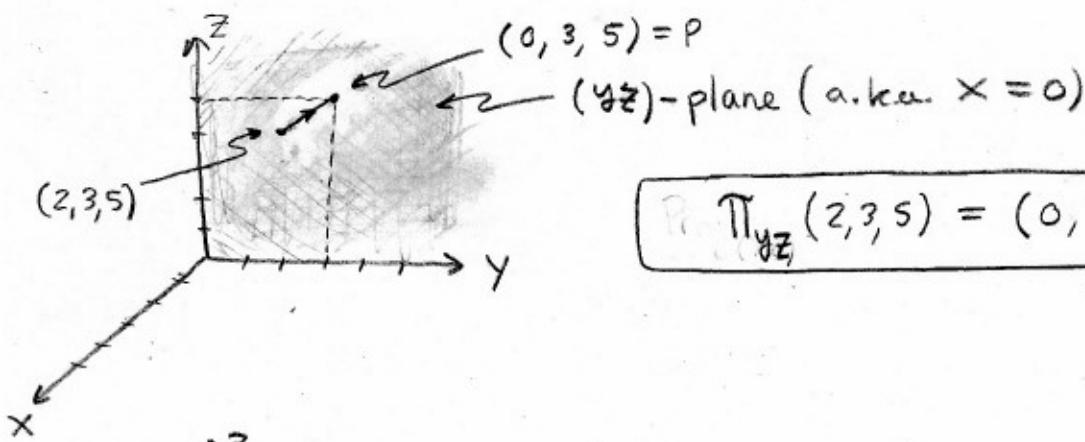
Problems taken from Stewart's "CALCULUS CONCEPTS & CONTEXTS" 3rd Ed.

§ 9.1 #4

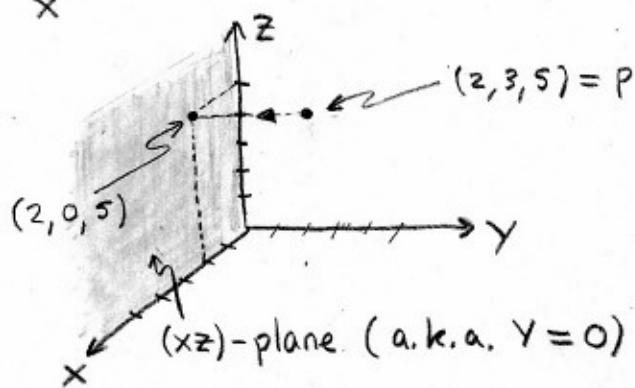
What are the projections of $(2, 3, 5)$ on the coordinate planes?



$$\Pi_{xy}(2, 3, 5) = (2, 3, 0)$$



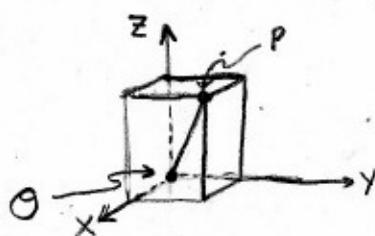
$$\Pi_{yz}(2, 3, 5) = (0, 3, 5)$$



$$\Pi_{xz}(2, 3, 5) = (2, 0, 5)$$

Draw the box with vertices $(0, 0, 0) \equiv O$, P and faces parallel to the coordinate planes. Then find the length of diagonal

the diagonal is the line segment \overline{OP}



$$\begin{aligned} |OP| &= \sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} \\ &= \sqrt{4+9+25} \\ &= \boxed{\sqrt{38}} = |OP| \end{aligned}$$

§9.1 #13 Show $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$ is a sphere.
Find its radius r and center (h, k, l) .

$$\left. \begin{array}{l} x^2 - 6x = (x-3)^2 - 9 \\ y^2 + 4y = (y+2)^2 - 4 \\ z^2 - 2z = (z-1)^2 - 1 \end{array} \right\} \text{Completed the square (x 3)}$$

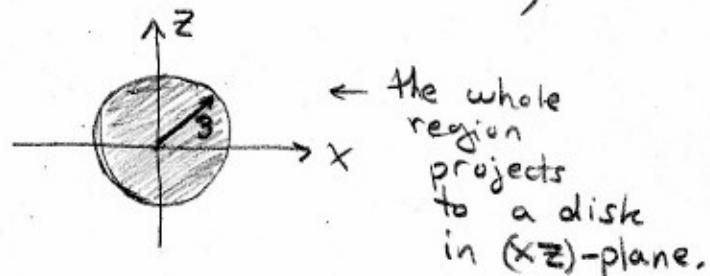
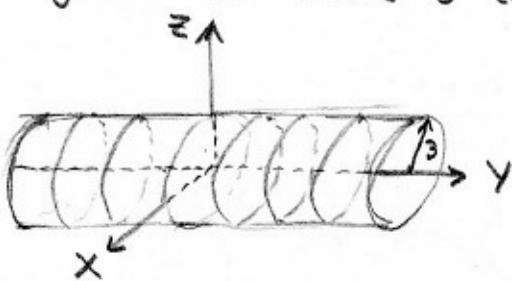
Thus we find, substituting the above into the given eqⁿ,

$$x^2 + y^2 + z^2 - 6x + 4y - 2z = (x-3)^2 + (y+2)^2 + (z-1)^2 - 14 = 11$$

$$\therefore \boxed{(x-3)^2 + (y+2)^2 + (z-1)^2 = 25}$$

This is a sphere centered at $(3, -2, 1)$ with $r = 5$.

§9.1 #37 Describe the region $x^2 + z^2 \leq 9$ in \mathbb{R}^3 . This is a cylinder of radius 3 centered on the y -axis. A sketch,



§9.1 #35 Find Σy^2 of all points equidistant from $A = (-1, 5, 3)$ and $B = (6, 2, -2)$

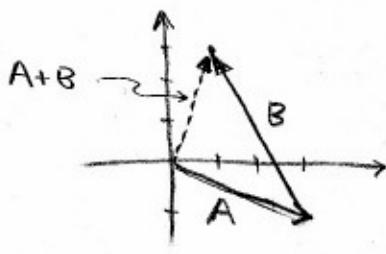
$$\begin{aligned} |AP| &= |BP| \\ \Rightarrow |AP|^2 &= |BP|^2 \\ \Rightarrow (x+1)^2 + (y-5)^2 + (z-3)^2 &= (x-6)^2 + (y-2)^2 + (z+2)^2 \\ \Rightarrow x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 &= \\ \Rightarrow x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 &= \end{aligned}$$

Now cancel the x^2, y^2, z^2 on both sides and collect like terms to find that

$$14x - 6y - 10z = 36 + 4 + 4 - 1 - 25 - 9 = 9$$

this is a plane with normal $\langle 14, -6, -10 \rangle$. (See §9.5 if you don't believe it)

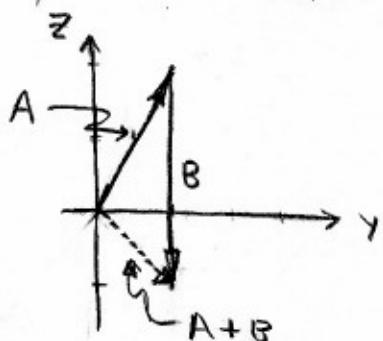
§9.2 #11 Let $A = \langle 3, -1 \rangle$ and $B = \langle -2, 4 \rangle$ then algebraically we can calculate $A+B = \langle 3, -1 \rangle + \langle -2, 4 \rangle = \langle 1, 3 \rangle$. Geometrically this is equivalent to the following picture,



$A+B$ is the "resultant" vector. Geometrically it is found by placing the "tail" of A at the origin and placing the "tail" of B at the "tip" of A . Then $A+B$ is identified as the vector formed from the origin to the "tip" of B . This is what is meant by "tip to tail" vector addition.

§9.2 #13 Let $A = \langle 0, 1, 2 \rangle$ and $B = \langle 0, 0, -3 \rangle$ find $A+B$.

We add components to obtain $A+B = \langle 0+0, 1+0, 2-3 \rangle = \langle 0, 1, -1 \rangle$.



- this problem only appears to be 3-dim'l. Actually it all happens in (yz) -plane.
 $x=0$ throughout.

§9.2 #17 Let $a = \hat{i} + 2\hat{j} - 3\hat{k}$ and $b = -2\hat{i} - \hat{j} + 5\hat{k}$.

$$a+b = \hat{i} + 2\hat{j} - 3\hat{k} - 2\hat{i} - \hat{j} + 5\hat{k} = \boxed{-\hat{i} + \hat{j} + 2\hat{k} = a+b}$$

which is easier to do via $a = \langle 1, 2, -3 \rangle$, $b = \langle -2, -1, 5 \rangle$ thus $a+b = \langle 1-2, 2-1, -3+5 \rangle = \langle -1, 1, 2 \rangle = -\hat{i} + \hat{j} + 2\hat{k}$. Next,

$$\begin{aligned} 2a + 3b &= 2\langle 1, 2, -3 \rangle + 3\langle -2, -1, 5 \rangle \\ &= \langle 2, 4, -6 \rangle + \langle -6, -3, 15 \rangle \\ &= \boxed{\langle -4, 1, 9 \rangle = -4\hat{i} + \hat{j} + 9\hat{k} = 2a + 3b} \end{aligned}$$

Next notice $a-b = \langle 1, 2, -3 \rangle - \langle -2, -1, 5 \rangle = \langle 3, 3, -8 \rangle$ thus

$$|a-b| = \sqrt{3^2 + 3^2 + (-8)^2} = \boxed{\sqrt{82} = |a-b|}$$

and of course,

$$|a| = \sqrt{1^2 + 2^2 + (-3)^2} = \boxed{\sqrt{14} = |a|}$$

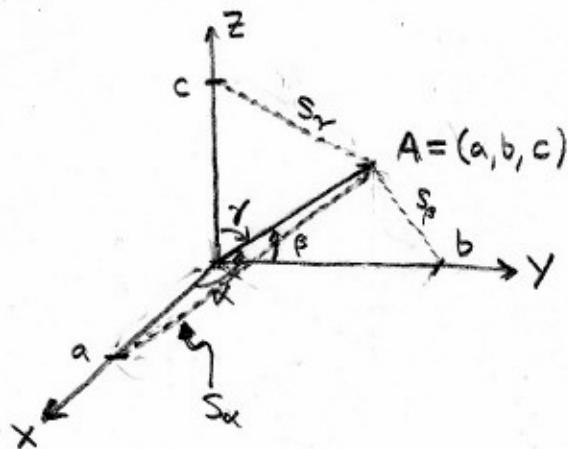
§9.2 #19 find unit-vector in direction of $B = 8\hat{i} - \hat{j} + 4\hat{k}$.

Notice that $\hat{B} = \frac{1}{|B|} B$ will be a vector of length one in direction of B . Observe $|B| = \sqrt{64+1+16} = \sqrt{81} = 9$ thus

$$\hat{B} = \frac{1}{9}(8\hat{i} - \hat{j} + 4\hat{k}) = \langle \frac{8}{9}, -\frac{1}{9}, \frac{4}{9} \rangle$$

Remark: I reserve the notation " $\hat{\cdot}$ " to indicate a vector of length one. This is why I put hats on \hat{i} , \hat{j} and \hat{k} . Generally any vector can be written as $A = |A|\hat{A}$, the $|A|$ gives us the magnitude while \hat{A} gives us the direction.

§9.2 #32 A vector $A \neq 0$ makes angles α , β , γ with the x , y , z axes respectively. Show that the "direction cosines" $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ satisfy the relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. For convenience of drawing,



I'll picture A in the 1st octant but the argument is more general than the picture.

$$\cos(\gamma) = \frac{c}{|A|}$$

$$\cos(\beta) = \frac{b}{|A|}$$

$$\cos(\alpha) = \frac{a}{|A|}$$

$$\begin{aligned}\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= \frac{a^2}{|A|^2} + \frac{b^2}{|A|^2} + \frac{c^2}{|A|^2} \\ &= \frac{a^2 + b^2 + c^2}{|A|^2} \\ &= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} = 1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma\end{aligned}$$

Remark: Next we learn $A \cdot B = |A||B| \cos \theta$. In particular this means $A \cdot \hat{i} = |A| \cos \alpha$, $A \cdot \hat{j} = |A| \cos \beta$, $A \cdot \hat{k} = |A| \cos \gamma$.

§9.3#1 Which of the following are sensible? Suppose a, b, c are vectors.

- (a.) $(a \cdot b) \cdot c \leftarrow a \cdot b$ is a scalar ($a \neq$), not sensible since dot-product needs two vector inputs.
- (b.) $(a \cdot b)c = (a \cdot b)\langle c_1, c_2, c_3 \rangle = \langle (a \cdot b)c_1, (a \cdot b)c_2, (a \cdot b)c_3 \rangle$
it's sensible, I show what it means explicitly above.
- (c.) $|a|(b \cdot c) = (\sqrt{a_1^2 + a_2^2 + a_3^2})(b_1c_1 + b_2c_2 + b_3c_3) \in \mathbb{R}$. Makes sense.
- (d.) $a \cdot (b+c) = a \cdot b + a \cdot c$, sensible.
- (e.) $a \cdot b + c$, not sensible, can't add a vector to a scalar.
- (f.) $|a| \cdot (b+c)$, not sensible $|a|$ is a scalar but dot-product takes vector inputs.

§9.3#6 $a = \langle s, 2s, 3s \rangle$ and $b = \langle t, -t, 5t \rangle$

$$\begin{aligned} a \cdot b &= \langle s, 2s, 3s \rangle \cdot \langle t, -t, 5t \rangle \\ &= st - 2st + 15st \\ &= 14st = a \cdot b \end{aligned}$$

§9.3#7 $a = \hat{i} - 2\hat{j} + 3\hat{k}$ and $b = 5\hat{i} + 9\hat{k}$

$$\begin{aligned} a \cdot b &= (\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (5\hat{i} + 9\hat{k}) \\ &= 5\cancel{\hat{i}} \cdot \cancel{\hat{i}}_1 + 9\cancel{\hat{i}} \cdot \cancel{\hat{k}}_0 - 10\cancel{\hat{j}} \cdot \cancel{\hat{i}}_0 - 18\cancel{\hat{j}} \cdot \cancel{\hat{k}}_0 + 15\cancel{\hat{k}} \cdot \cancel{\hat{i}}_1 + 27\cancel{\hat{k}} \cdot \cancel{\hat{k}}_1 \\ &= 5 + 27 \\ &= 32 = a \cdot b \end{aligned}$$

Remark: the calculation in #6 is clearly easier. However, the more algebraic approach in #7 is useful in certain contexts, for example when a mixture of coordinate systems are in play.

§9.3#17a Determine if the vectors are orthogonal, parallel, or neither.

- (a.) $u \cdot v = \langle -5, 3, 7 \rangle \cdot \langle 6, -8, 2 \rangle = -30 - 24 + 14 = -40 \neq 0$
these are not orthogonal. Note $|u| = \sqrt{25+9+49} = \sqrt{83}$
and $|v| = \sqrt{36+64+4} = \sqrt{104}$ then $|u||v| = \sqrt{(104)(83)} = \sqrt{8932}$
if u was parallel to v then $u \cdot v = \pm |u||v|$ yet we
find $u \cdot v = -40 \neq \pm \sqrt{8932}$. So $u \neq v$ are neither orthogonal or parallel.

§9.3 # 18c Let $u = \langle a, b, c \rangle$ and $v = \langle -b, a, 0 \rangle$.

$$u \cdot v = \langle a, b, c \rangle \cdot \langle -b, a, 0 \rangle = -ab + ab + 0 = 0$$

Thus $u \cdot v = 0$ hence u is orthogonal to v .

(H6)

§9.3 # 26 Let $a = \hat{i} + \hat{j} + \hat{k} = \langle 1, 1, 1 \rangle$ and $b = \hat{i} - \hat{j} + \hat{k} = \langle 1, -1, 1 \rangle$
then find the scalar and vector projections of b onto a

$$\text{comp}_a(b) = b \cdot \hat{a} = \langle 1, -1, 1 \rangle \cdot \left[\frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right] = \frac{1}{\sqrt{3}} (1 - 1 + 1) = \boxed{\frac{1}{\sqrt{3}}}$$

we find the "scalar projection of b onto a " is $\boxed{\text{comp}_a(b) = \frac{1}{\sqrt{3}}}$

In other words the component of b in the a -direction is $\frac{1}{\sqrt{3}}$.

$$\text{proj}_a(b) = (\text{comp}_a(b)) \hat{a} = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) = \frac{1}{3} \langle 1, 1, 1 \rangle.$$

The projection of b onto the a -axis is the vector projection
of b onto a , in particular $\boxed{\text{proj}_a(b) = \frac{1}{3} \langle 1, 1, 1 \rangle}$

Remark: the concepts of comp_a & proj_a are used throughout this course. Usually its implicitly in the case $a = \hat{i}, \hat{j}$ or \hat{k} then the $\text{comp}_{\hat{i}}, \text{comp}_{\hat{j}}, \text{comp}_{\hat{k}}$ just give x, y, z components respectively. While $\text{proj}_{\hat{i}}, \text{proj}_{\hat{j}}, \text{proj}_{\hat{k}}$ give us the part of the vector that goes in the x, y or z directions respectively.

§9.3 # 30 Consider vectors $a, b \neq 0$. When is $\text{comp}_a(b) = \text{comp}_b(a)$? Consider towards answering the question the following equality

$$\begin{aligned} b \cdot \hat{a} = a \cdot \hat{b} &\iff (|b| \hat{b}) \cdot \hat{a} = (|a| \hat{a}) \cdot \hat{b} \\ &\iff |b| \hat{b} \cdot \hat{a} = |a| \hat{a} \cdot \hat{b} \\ &\iff |b| \hat{a} \cdot \hat{b} = |a| \hat{a} \cdot \hat{b} \\ &\iff [|b| - |a|] \hat{a} \cdot \hat{b} = 0 \\ &\iff |b| = |a| \quad \text{or} \quad \hat{a} \cdot \hat{b} = 0 \end{aligned}$$

\iff a is orthogonal to b (no restriction on $|a|, |b|$)
 a not orthogonal to b and have same length.

§9.3 #30b If $a, b \neq 0$ are vectors then when is $\text{proj}_a(b) = \text{proj}_b(a)$? Consider the following equality as in part (a),

$$\begin{aligned}
 (b \cdot \hat{a})\hat{a} &= (a \cdot \hat{b})\hat{b} \Leftrightarrow |b|(\hat{b} \cdot \hat{a})\hat{a} = |a|(\hat{a} \cdot \hat{b})\hat{b} \\
 &\Leftrightarrow \hat{a} \cdot \hat{b} [|b|\hat{a} - |a|\hat{b}] = 0 \\
 &\Leftrightarrow \hat{a} \cdot \hat{b} = 0 \text{ or } |b|\hat{a} = |a|\hat{b} \\
 &\Leftrightarrow \boxed{a \text{ orthogonal to } b \text{ (no restriction on lengths)}} \\
 &\quad \text{or } \hat{a} = \frac{|a|}{|b|} \hat{b} \text{ but since } |\hat{a}| = |\hat{b}| = 1 \\
 &\quad \text{this tells us } |a|/|b| = 1 \therefore |a| = |b| \\
 &\quad \text{hence } \hat{a} = \hat{b} \Rightarrow \boxed{a = b}.
 \end{aligned}$$

Remark: # 30 seems like a nice test question... hee hee hee...

§9.3 #31 Let $F = \langle 10, 18, -6 \rangle$ be a constant force field which moves an object from $(2, 3, 0)$ to $(4, 9, 15)$.

$$\begin{aligned}
 W &= F \cdot (\Delta x) = \langle 10, 18, -6 \rangle \cdot \langle 4-2, 9-3, 15-0 \rangle \\
 &= \langle 10, 18, -6 \rangle \cdot \langle 2, 6, 15 \rangle \\
 &= 20 + 108 - 90 \\
 &= \boxed{38 = W} \quad \text{with units } \boxed{W = 38 \text{ J}}
 \end{aligned}$$

• assuming $[F] = \text{Newtons}$, $[\Delta x] = \text{meters}$, we'll avoid units in this course, I'm happy to tell you how to add them correctly if you're interested, it's just more writing and I think it distracts from the math here.

§9.3 #43 Recall $A \cdot B = |A||B|\cos\theta$ and $-1 \leq \cos\theta \leq 1$ thus $|\cos\theta| \leq 1$

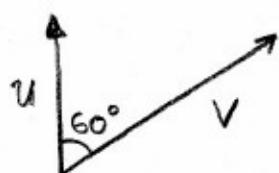
$$|A \cdot B| = ||A||B||\cos\theta|| = |A||B||\cos\theta| \leq |A||B| \text{ using } |\cos\theta| \leq 1.$$

Thus $|A \cdot B| \leq |A||B|$, this is the Cauchy-Schwarz Inequality. Notice 1.1 has two meanings: 1.) absolute value 2.) vector length.

Remark: the Cauchy-Schwarz Inequality holds for many abstract systems of math, in fact it is one of the defining conditions for a Hilbert Space.

§9.4 #2 Find $|u \times v|$ and $\hat{u} \times \hat{v}$ (the unit vector in $u \times v$ direction) given that

$$|u| = 5 \text{ and } |v| = 10 \text{ and } \theta = 60^\circ$$



$$|u \times v| = |u||v| \sin \theta = (5)(10) \sin 60^\circ = 50 \frac{\sqrt{3}}{2} = 25\sqrt{3}$$

point fingers in direction of u then curl them into the direction of v (use right hand). Your thumb will point into the page.

$u \times v$ goes into the page

§9.4 #11 Let $a = \langle 3, 2, 4 \rangle$ and $b = \langle 1, -2, -3 \rangle$

$$\begin{aligned} a \times b &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \hat{i} \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \\ &= \hat{i}(-6+8) - \hat{j}(9-4) + \hat{k}(-6-2) \\ &= \boxed{2\hat{i} + 13\hat{j} - 8\hat{k} = \langle 2, 13, -8 \rangle = a \times b} \end{aligned}$$

Notice $a \cdot (a \times b) = \langle 3, 2, 4 \rangle \cdot \langle 2, 13, -8 \rangle = 6 + 26 - 32 = 0$
and $b \cdot (a \times b) = \langle 1, -2, -3 \rangle \cdot \langle 2, 13, -8 \rangle = 2 - 26 + 24 = 0$. } $a \not\parallel b$
to $a \times b$.

Alternatively we could use the algebra of cross products

$$\begin{aligned} a \times b &= (3\hat{i} + 2\hat{j} + 4\hat{k}) \times (\hat{i} - 2\hat{j} - 3\hat{k}) \\ &= 3\hat{i} \times \hat{i} - 6\hat{i} \times \hat{j} - 9\hat{i} \times \hat{k} + 2\hat{j} \times \hat{i} - 4\hat{j} \times \hat{j} - 6\hat{j} \times \hat{k} \\ &\quad + 4\hat{k} \times \hat{i} - 8\hat{k} \times \hat{j} - 12\hat{k} \times \hat{k} \\ &= -6\hat{k} + 9\hat{j} - 2\hat{k} - 6\hat{i} + 4\hat{j} + 8\hat{i} \\ &= \langle 8-6, 9+4, -6-2 \rangle \\ &= \boxed{\langle 2, 13, -8 \rangle} \end{aligned}$$

BONUS MATERIAL

Using Einstein's index notation we saw

$$a \times b = (\epsilon_{ijk} a_i b_j) e_k$$

$$a \cdot c = a_k c_k$$

for $a = a_k e_k$ and $b = b_k e_k$ and $c = c_k e_k$

$$(a \times b) \cdot a = \epsilon_{ijk} a_i b_j a_k$$

$$= -\epsilon_{kji} a_k b_j a_i = - (a \times b) \cdot a \therefore \boxed{(a \times b) \cdot a = 0}$$

this proves it for arbitrary a, b !

$$\begin{array}{lcl} \hat{i} \times \hat{i} & = & \hat{k} \\ \hat{i} \times \hat{j} & = & \hat{k} \\ \hat{j} \times \hat{i} & = & \hat{-k} \\ \hat{k} \times \hat{i} & = & \hat{j} \\ \hat{j} \times \hat{k} & = & \hat{i} \\ \hat{k} \times \hat{j} & = & \hat{-i} \\ \hat{i} \times \hat{k} & = & \hat{-j} \end{array}$$

§9.4 #21 $V = |a \cdot (b \times c)|$ so given a, b, c we calculate,

$$b \times c = \langle 0, 1, 2 \rangle \times \langle 4, -2, 5 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= 9\hat{i} + 8\hat{j} - 4\hat{k}$$

$$= \langle 9, 8, -4 \rangle$$

$$a \cdot (b \times c) = \langle 6, 3, -1 \rangle \cdot \langle 9, 8, -4 \rangle$$

$$= 54 + 24 + 4$$

$$= \boxed{82 = V}$$

Remark: Exercises 30 & 32 can be proved brute-force. I offer an alternative approach that requires some initial investment of notational work but in the end greatly shortens abstract calculations. In #11 I gave a short taste of these ideas.

Notation and the Einstein Index Convention

$$e_1 = \hat{i}, e_2 = \hat{j}, e_3 = \hat{k}$$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if } i, j, k \text{ are not distinct} \\ 1 & \text{if } (i, j, k) \text{ are } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2) \\ -1 & \text{if } (i, j, k) \text{ are } (3, 2, 1), (2, 1, 3) \text{ or } (1, 3, 2) \end{cases}$$

The δ_{ij} is the Kronecker Delta, the ϵ_{ijk} is the antisymmetric symbol. Some useful identities are

$\delta_{ij} = \delta_{ji}$	$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{teji} = -\epsilon_{ikj}$
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one may then employ the "repeated index" notation

$$A_i B_i \equiv A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$A_{jk} B_j = A_{1k} B_1 + A_{2k} B_2 + A_{3k} B_3$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Usually in math we'd write $\sum_{i=1}^3$ or $\sum_{j=1}^3$ or $\sum_{k=1}^3$ in the above but it is a convenient shorthand to omit the $i=1$ as indicated here.

Einstein Index Convention Short-cuts Continued

(H10)

We observe that we can write dot and cross products of $A = A_i e_i$ and $B = B_j e_j$ as follows

$$\begin{aligned}
 A \cdot B &= (A_i e_i) \cdot (B_j e_j) \\
 &= A_i B_j e_i \cdot e_j \\
 &= A_i B_j \delta_{ij} \quad : \text{notice } e_i \cdot e_j = \delta_{ij} \text{ think about it.} \\
 &= \boxed{A_i B_i} = A \cdot B \quad \text{nice formula.}
 \end{aligned}$$

Next observe that $e_i \times e_j = \epsilon_{ijk} e_k$. For example we have $e_1 \times e_2 = \epsilon_{12k} e_k = \epsilon_{123} e_3 = e_3$ ($\hat{i} \times \hat{j} = \hat{k}$) thus we may calculate

$$\begin{aligned}
 A \times B &= (A_i e_i) \times (B_j e_j) \\
 &= A_i B_j (e_i \times e_j) \\
 &= \boxed{A_i B_j \epsilon_{ijk} e_k} = A \times B \quad \text{nice formula}
 \end{aligned}$$

We can read-off that $(A \times B)_k = A_i B_j \epsilon_{ijk}$. Finally for future convenience I alert the interested student to some über-useful formulas (which I invite you to verify case-wise, its tedious but not hard)

Über-Lemmas:

- (i) $\epsilon_{ijk} \epsilon_{mnl} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$
- (ii) $\epsilon_{ijk} \epsilon_{mjk} = 2\delta_{im}$
- (iii) If $A_{ij} = -A_{ji}$ and $S_{ij} = S_{ji}$ for all i, j then $A_{ij} S_{ij} = 0$. "Symmetric kills Antisymmetric"
- (iv) $B_{ij} \delta_{jk} = B_{ik}$, Kronecker δ_{jk} makes $j = k$.

• Now the toys are unwrapped, let us play with them ↴

§9.4 #29 Ok, well we don't need my toys yet. This follows from elementary distributive property of \times plus $A \times B = -B \times A$, (HII)

$$\begin{aligned}
 (A - B) \times (A + B) &= A \times (A + B) - B \times (A + B) \\
 &= A \times A + A \times B - B \times A - B \times B \\
 &= 0 + A \times B - (-A \times B) + 0 \\
 &= \underline{2A \times B} //.
 \end{aligned}$$

Notice $A \times A = -A \times A \therefore 2A \times A = 0 \therefore A \times A = 0$. We should always keep this in mind.

§9.4 #30 this is considerably longer via brute-force,

$$\begin{aligned}
 A \times (B \times C) &= A_i (B \times C)_j \epsilon_{ijk} e_k \\
 &= A_i \epsilon_{mni} B_m C_n \epsilon_{ijk} e_k \quad \rightarrow \epsilon_{ijk} = -\epsilon_{ikj} \\
 &= -A_i B_m C_n \epsilon_{mni} \epsilon_{ikj} e_k \quad \rightarrow \text{using (i.) of} \\
 &= -A_i B_m C_n [S_{mi} S_{nk} - S_{mk} S_{ni}] e_k \quad \leftarrow \text{uber-lemma.} \\
 &= [-A_m B_m C_k + A_n B_k C_n] e_k \quad \rightarrow \text{distributing and} \\
 &= (A_n C_n) B_k e_k - (A_m B_m) C_k e_k \quad \rightarrow \text{using (iv.) of} \\
 &= \underline{(A \cdot C) B - (A \cdot B) C} //.
 \end{aligned}$$

§9.4 #31 Consider if we add #30 together for "cyclic permutations" of A, B, C we find

$$+ \begin{cases} A \times (B \times C) = (A \cdot C) B - (A \cdot B) C \\ B \times (C \times A) = (B \cdot A) C - (B \cdot C) A \\ C \times (A \times B) = (C \cdot B) A - (C \cdot A) B \end{cases}$$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0 \quad -(***)$$

since ① cancels due to $A \cdot C = C \cdot A$ and likewise for $A \cdot B = B \cdot A$ and $C \cdot B = B \cdot C$ yield cancellations ② & ③.

Remark: (***) is the Jacobi Identity. There are abstract algebras called "Lie Algebras" that satisfy this condition.

(H12)

§9.4 #32 If you prefer brute-force then feel free to do it that way.

$$\begin{aligned}
 (A \times B) \cdot (C \times D) &= (\sum_{ijk} A_i B_j)(\sum_{mnl} C_m D_n) \\
 &= \sum_{ijk} \sum_{mnl} A_i B_j C_m D_n \\
 &= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) A_i B_j C_m D_n \\
 &= A_m B_n C_m D_n - A_n B_m C_m D_n \\
 &= (A_m C_m)(B_n D_n) - (A_n D_n)(B_m C_m) \\
 &= (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C) \\
 &= \boxed{\begin{vmatrix} A \cdot C & B \cdot C \\ A \cdot D & B \cdot D \end{vmatrix}} //.
 \end{aligned}$$

§9.5 #3 Find eq's of line through $(-2, 4, 10)$ and parallel to $\langle 3, 1, -8 \rangle$,

$$r(t) = \langle -2, 4, 10 \rangle + t \langle 3, 1, -8 \rangle = \boxed{\langle 3t-2, t+4, -8t+10 \rangle} = r(t)$$

A.k.a. $x = 3t-2, y = t+4$ and $z = -8t+10$.

§9.5 #5 find line through $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$. Notice the normal to this plane is $N = \langle 1, 3, 1 \rangle$. To say line is perpendicular to a plane is to say the line is parallel to the normal.

$$r(t) = \langle 1, 0, 6 \rangle + t \langle 1, 3, 1 \rangle = \boxed{\langle t+1, 3t, t+6 \rangle} = r(t)$$

parametric eq's are $x = t+1, y = 3t$ and $z = t+6$.

WARNING: in my notes & lecture $z = 3y$ - Sorry, but not so sorry to change. ☺.