

§ 11.2 #5 Find the limit, if it exists.

$$\lim_{(x,y) \rightarrow (5,-2)} (x^5 + 4x^3y - 5xy^2) = 5^5 - 45^3(-2) - 5(5)(4) = 2025$$

this is the simple case where the function is continuous at the limit point. This means $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$. Usually we cannot evaluate $f(a,b)$ and therein lies the difficulty, and the utility of the limit. Anyway, functions are continuous where they make sense.

§ 11.2 #9 Find the limit, if it exists

$$\lim_{(x,y) \rightarrow (0,0)} \left(\frac{xy \cos(y)}{3x^2 + y^2} \right) \text{ well notice } \frac{xy \cos(y)}{3x^2 + y^2} \Big|_{(0,0)} = \frac{0}{0}$$

which is indeterminate, we could get no value, $\pm\infty$ or even a finite #. If the limit exists then for all possible continuous paths to zero we should have the same limit. Approach along x -axis $(x,0) \rightarrow (0,0)$,

$$\lim_{(x,0) \rightarrow (0,0)} \left(\frac{xy \cos(y)}{3x^2 + y^2} \right) = \lim_{x \rightarrow 0} \left(\frac{0}{3x^2 + 0} \right) = 0.$$

Now approach $(0,0)$ along $y=x$,

$$\lim_{(x,x) \rightarrow (0,0)} \left(\frac{x^2 \cos(x)}{4x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{\cos(x)}{4} \right) = \frac{\cos(0)}{4} = \frac{1}{4}$$

Thus we find different limits along different paths approaching $(0,0)$ therefore the limit does not exist

Remark: there are more subtle cases I'll not emphasize here.

In practice we almost never take such limits in applications. You might look at the online calc III videos for a few more sophisticated examples if you enjoy these. Higher dim'l limits require a bit more ingenuity than the 1-dim'l limits.

$$\text{§ 11.3 #14} \quad f(x,y) = x^5 + 3x^3y^2 + 3xy^4$$

$$\frac{\partial f}{\partial x} = 5x^4 + 9x^2y^2 + 3y^4$$

$$\frac{\partial f}{\partial y} = 6x^3y + 12xy^3$$

$$\text{§ 11.3 #16} \quad f(x,y) = y \ln(x) = z$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [y \ln(x)] = y \frac{\partial}{\partial x} [\ln(x)] = \frac{y}{x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [y \ln(x)] = \ln(x) \frac{\partial}{\partial y} [y] = \ln(x).$$

$$\text{§ 11.3 #18} \quad f(x,y) = x^y$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^y) = yx^{y-1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^y) = \frac{\partial}{\partial y} (e^{\ln(x^y)}) = \frac{\partial}{\partial y} (e^{y \ln(x)}) = e^{y \ln(x)} \cdot \frac{\partial}{\partial y} (y \ln(x))$$

$$\Rightarrow \frac{\partial f}{\partial y} = e^{y \ln(x)} \cdot \ln(x) = \boxed{\ln(x) x^y}$$

(reversing steps to
convert back to x^y)

- (this is just rederiving $\frac{d}{du}(a^u) = \ln(a) a^u$, you could use that)
directly if you see it. I say remember little, derive a lot.)

$$\text{§ 11.3 #20} \quad f(s,t) = st^2/(s^2+t^2)$$

$$\frac{\partial f}{\partial s} = t^2 \frac{\partial}{\partial s} \left[\frac{s}{s^2+t^2} \right] = t^2 \left[\frac{s^2+t^2 - 2s \cdot s}{(s^2+t^2)^2} \right] = \boxed{\frac{t^2(t^2-s^2)}{(s^2+t^2)^2}}$$

$$\frac{\partial f}{\partial t} = s \frac{\partial}{\partial t} \left[\frac{t^2}{s^2+t^2} \right] = s \left[\frac{2t(s^2+t^2) - 2t \cdot t^2}{(s^2+t^2)^2} \right] = \boxed{\frac{2ts^3}{(s^2+t^2)^2}}$$

$$\text{§ 11.3 #24} \quad f(x,y) = \int_y^x \cos(t^2) dt = F(x) - F(y) \text{ where } F'(u) = \cos(u^2).$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left[\int_y^x \cos(t^2) dt \right] = \frac{\partial}{\partial x} (F(x) - F(y)) = \frac{\partial F}{\partial x}(x) = \cos(x^2).$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\int_y^x \cos(t^2) dt \right] = \frac{\partial}{\partial y} (F(x) - F(y)) = \frac{\partial F}{\partial y}(y) = \cos(y^2).$$

§11.3 #19 Let $w = \sin\alpha \cos\beta$

$$\frac{\partial w}{\partial \alpha} = \frac{\partial}{\partial \alpha} [\sin\alpha \cos\beta] = \cos\beta \frac{\partial}{\partial \alpha} [\sin\alpha] = \boxed{\cos\beta \cos\alpha}$$

$$\frac{\partial w}{\partial \beta} = \frac{\partial}{\partial \beta} [\sin\alpha \cos\beta] = \sin\alpha \frac{\partial}{\partial \beta} [\cos\beta] = \boxed{-\sin\alpha \sin\beta}$$

§11.3 #33 $u = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. Let $1 \leq k \leq n$ then,

$$\begin{aligned}\frac{\partial u}{\partial x_k} &= \frac{\partial}{\partial x_k} \left[\sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \right] \\ &= \frac{1}{2\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}} \frac{\partial}{\partial x_k} [x_1^{2^0} + x_2^{2^0} + \dots + \overset{2x_k}{x_k^{2^0}} + \dots + x_n^{2^0}] \\ &= \boxed{\frac{x_k}{\sqrt{x_1^2 + \dots + x_n^2}}}\end{aligned}$$

§11.3 #37 Let $f(x, y, z) = x/(y+z)$ find $f_z(3, 2, 1) = \frac{\partial f}{\partial z} \Big|_{(3, 2, 1)}$

$$\frac{\partial f}{\partial z} \Big|_{(3, 2, 1)} = \frac{-x}{(y+z)^2} \Big|_{(3, 2, 1)} = \frac{-3}{(2+1)^2} = \frac{-3}{9} = \boxed{-\frac{1}{3}}$$

§11.3 #41 find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ via implicit differentiation.

Suppose that $x^2 + y^2 + z^2 = 3xyz$, assume $z = z(x, y)$,

$$\frac{\partial}{\partial x} [x^2 + y^2 + z^2] = \frac{\partial}{\partial x} [3xyz]$$

$$2x + 2z \frac{\partial z}{\partial x} = 3xy \frac{\partial z}{\partial x} + 3yz$$

$$\frac{\partial z}{\partial x} (2z - 3xy) = 3yz - 2x \quad \therefore$$

$$\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2z - 3xy}$$

Likewise

$$2y + 2z \frac{\partial z}{\partial y} = 3xz \frac{\partial z}{\partial y} + 3xz$$

$$\frac{\partial z}{\partial y} (2z - 3xz) = 3xz - 2y \quad \therefore$$

$$\frac{\partial z}{\partial y} = \frac{3xz - 2y}{2z - 3xz}$$

§11.3 #43 $x - z = \tan^{-1}(yz)$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$,

$$\frac{\partial}{\partial x}[x - z] = \frac{\partial}{\partial x}[\tan^{-1}(yz)]$$

$$1 - \frac{\partial z}{\partial x} = \frac{1}{1+(yz)^2} \frac{\partial}{\partial x}[yz] = \frac{y}{1+(yz)^2} \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} \left[\frac{y}{1+(yz)^2} + 1 \right] = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{\frac{y}{1+(yz)^2} + 1} = \boxed{\frac{1+y^2z^2}{y+1+y^2z^2}} = \frac{\partial z}{\partial x}$$

Next take $\frac{\partial z}{\partial y}$,

$$\begin{aligned} -\frac{\partial z}{\partial y} &= \frac{1}{1+(yz)^2} \frac{\partial}{\partial y}(yz) \\ &= \frac{1}{1+y^2z^2} [z + y \frac{\partial z}{\partial y}] \end{aligned}$$

$$\Rightarrow \frac{\partial z}{\partial y} \left[\frac{y}{1+y^2z^2} + 1 \right] = \frac{-z}{1+y^2z^2}$$

$$\Rightarrow \frac{\partial z}{\partial y} (y + 1 + y^2z^2) = -z$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-z}{y+1+y^2z^2}}$$

§11.3 #45 find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following,

(a.) $z = f(x) + g(y)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x) + g(y)] = \frac{\partial f}{\partial x} = \frac{df}{dx} : \text{since } f \text{ is frct. of } x \text{ only.}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x) + g(y)] = \frac{\partial g}{\partial y} = \frac{dg}{dy} : \text{since } g \text{ is frct. of } y \text{ only.}$$

(b.) $z = f(x+y)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x+y)] = f'(x+y) \frac{\partial}{\partial x}[x+y]^1 = f'(x+y).$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x+y)] = f'(x+y) \frac{\partial}{\partial y}[x+y]^1 = f'(x+y).$$

§11.3 #50 find 2nd partials of $z = y \tan(2x)$

$$\frac{\partial z}{\partial y} = \tan(2x) \text{ then } \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} [\tan(2x)] = 0.$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2y \sec^2(2x) \text{ then } \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} [2y \sec^2(2x)] \\ &= 2y \cdot 2\sec(2x) \cdot \frac{\partial}{\partial x} [\sec(2x)] \\ &= 4y \sec(2x) \sec(2x) \tan(2x) \cdot 2 \\ &= 8y \sec^2(2x) \tan(2x) = \boxed{\frac{\partial^2 z}{\partial x^2}}\end{aligned}$$

You can calculate the remaining 2nd order partial derivative which is $\boxed{z_{xy}}$.

§11.3 #56 $f(x, t) = x^2 e^{-ct}$ where C is a constant.

$$\begin{aligned}f_{ttt} &= \frac{\partial}{\partial t} \frac{\partial}{\partial t} \frac{\partial}{\partial t} (x^2 e^{-ct}) \\ &= x^2 \frac{\partial}{\partial t} \frac{\partial}{\partial t} (-ce^{-ct}) \\ &= x^2 \frac{\partial}{\partial t} (c^2 e^{-ct}) \\ &= \boxed{-x^2 c^3 e^{-ct} = f_{ttt}}\end{aligned}$$

$$\begin{aligned}f_{txx} &= \frac{\partial}{\partial t} \frac{\partial}{\partial x} \frac{\partial}{\partial x} (x^2 e^{-ct}) \\ &= \frac{\partial}{\partial t} \frac{\partial}{\partial x} (2x e^{-ct}) \\ &= \frac{\partial}{\partial t} (2e^{-ct}) \\ &= \boxed{-2c e^{-ct} = f_{txx}}\end{aligned}$$

§11.3 #65 Verify $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ solves $u_{xx} + u_{yy} + u_{zz} = 0$.
 Notice from #33 we can already state that
 if $W \equiv \sqrt{x^2+y^2+z^2}$ then $W_x = \frac{x}{W}$

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left[\frac{1}{W}\right]$$

$$= -\frac{1}{W^2} \frac{\partial W}{\partial x}$$

$$= -\frac{1}{W^2} \frac{x}{W}$$

$$= \frac{-x}{W^3}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{-x}{W^3}\right]$$

$$= \frac{-1W^3 + x \cdot 3W^2 W_x}{W^6}$$

$$= \frac{-W^3 + 3x^2 W}{W^6} = u_{xx} = \frac{-W^2 + 3x^2}{W^5}$$

Likewise, u_{yy} and u_{zz} have same form with $x \rightarrow y$ or z ,

$$u_{xx} + u_{yy} + u_{zz} = -\frac{W^2 + 3x^2}{W^5} + \frac{-W^2 + 3y^2}{W^5} + \frac{-W^2 + 3z^2}{W^5}$$

$$= \frac{-3W^2 + 3(x^2 + y^2 + z^2)}{W^5}$$

$$= \frac{-3(x^2 + y^2 + z^2) + 3(x^2 + y^2 + z^2)}{W^5}$$

$$= 0. \quad \therefore u \text{ solves } u_{xx} + u_{yy} + u_{zz} = \nabla^2 u = 0.$$

We'll explain later.

Remark: I'm pretty-sure that introducing W makes life easier here.

§11.5#2 Let $z = x \ln(x + 2y)$, $x = \sin t$, $y = \cos t$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= \left[\ln(x + 2y) + \frac{x}{x+2y} \right] \frac{dx}{dt} + \left[\frac{2x}{x+2y} \right] \frac{dy}{dt} \\ &= \boxed{\left[\ln(\sin t + 2\cos t) + \frac{\sin t}{\sin t + 2\cos t} \right] \cos t - \frac{2\sin^2 t}{\sin t + 2\cos t}}\end{aligned}$$

Remark: Because explicit formulas for $x(t)$ and $y(t)$ were given here one could also first substitute in $X(t)$ and $Y(t)$ to obtain $z(t) = \sin t \ln(\sin t + 2\cos t)$ then differentiate ala ma 141 to get same answer. It is better to practice the method using the chain-rule because I might not give you explicit formulas for $x(t)$ and $y(t)$.

§11.5#7 Let $z = e^r \cos \theta$ and suppose $r = st$, $\theta = \sqrt{s^2 + t^2}$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial s} \\ &= e^r \cos \theta \cdot \frac{\partial}{\partial s}(st) - e^r \sin \theta \frac{\partial}{\partial s}(\sqrt{s^2 + t^2}) \\ &= te^r \cos \theta - e^r \sin \theta \frac{s}{\sqrt{s^2 + t^2}} \quad \text{but remember that} \\ &= \boxed{e^{st} \left\{ t \cos \sqrt{s^2 + t^2} - (\sin \sqrt{s^2 + t^2}) \frac{s}{\sqrt{s^2 + t^2}} \right\}} = \frac{\partial z}{\partial s}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial t} \\ &= e^r \cos \theta \cdot s - e^r \sin \theta \frac{t}{\sqrt{s^2 + t^2}} \\ &= \boxed{e^{st} \left[s \cos \sqrt{s^2 + t^2} - (\sin \sqrt{s^2 + t^2}) \frac{t}{\sqrt{s^2 + t^2}} \right]} = \frac{\partial z}{\partial t}\end{aligned}$$

- the text's answer key is lazy on this problem.

§11.5 #10 Let $W(s, t) = F(u(s, t), v(s, t))$

where F, u, v are differentiable and

$$u(1, 0) = 2$$

$$v(1, 0) = 3$$

$$F_u(2, 3) = -1$$

$$u_s(1, 0) = -2$$

$$v_s(1, 0) = 5$$

$$F_v(2, 3) = 10$$

$$u_t(1, 0) = 6$$

$$v_t(1, 0) = 4$$

Find $W_s(1, 0)$ and $W_t(1, 0)$ using given data,

$$\frac{\partial W}{\partial s} = \frac{\partial}{\partial s}[F(u(s, t), v(s, t))] = \frac{\partial F}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial s} = F_u u_s + F_v v_s$$

$$\begin{aligned} W_s(1, 0) &= \left. \frac{\partial W}{\partial s} \right|_{(1, 0)} = F_u(u(1, 0), v(1, 0)) u_s(1, 0) + F_v(u(1, 0), v(1, 0)) v_s(1, 0) \\ &= F_u(2, 3) \cdot (-2) + F_v(2, 3) \cdot (5) \\ &= (-1)(-2) + (10)(5) \\ &= \boxed{52 = W_s(1, 0)} \end{aligned}$$

Likewise,

$$\begin{aligned} W_t(1, 0) &= F_u(2, 3) u_t(1, 0) + F_v(2, 3) v_t(1, 0) \\ &= (-1)(6) + (10)(4) \\ &= \boxed{34 = W_t(1, 0)} \end{aligned}$$

§11.5 #17 I'll find $\frac{\partial z}{\partial u}$ for $z = x^2 + xy^3$ with $x = uv^2 + w^3$
 $y = u + ve^w$
when $u=2, v=1$ and $w=0$,

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial}{\partial u}(z(x(u, v, w), y(u, v, w))) \\ &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ &= (2x + y^3)v^2 + (3xy^2) \cdot 1 \end{aligned}$$

I leave the remainder of this problem for you to work out.

Notice $x(2, 1, 0) = 2(1) + 0^3 = 2$ and $y(2, 1, 0) = 2 + 1 \cdot e^0 = 3$.

$$\left. \frac{\partial z}{\partial u} \right|_{(2, 1, 0)} = \underbrace{(2(2) + 3^3)}_{31} 1^2 + \underbrace{3(2)(3)}_{54}^2 = \boxed{85 = \left. \frac{\partial z}{\partial u} \right|_{(2, 1, 0)}}$$

§11.5 #24 Given $F(x, y) = 0$ we can show that

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}. \text{ Let } F(x, y) = \sin(x)\cos(y) - \sin(x) - \cos(y).$$

Calculate them,

$$\frac{\partial F}{\partial x} = \cos(x)\cos(y) - \cos(x) \quad \frac{\partial F}{\partial y} = -\sin(x)\sin(y) + \sin(y)$$

Therefore,

$$\frac{dy}{dx} = \frac{-\cos(x)\cos(y) + \cos(x)}{-\sin(x)\sin(y) + \sin(y)} = \boxed{\frac{\cos(x)}{\sin(y)} \left[\frac{1 - \cos(y)}{1 - \sin(y)} \right] = \frac{dy}{dx}}$$

§11.5 #27 Given $F(x, y, z) = 0$ we can show that if $z = f(x, y)$

then $\frac{\partial z}{\partial x} = -F_x/F_z$ and $\frac{\partial z}{\partial y} = -F_y/F_z$. Begin by noting

$$F(x, y, z) = \tan^{-1}(yz) + z - x \text{ thus calculate,}$$

$$F_x = -1$$

$$F_y = \frac{z}{1+y^2z^2}$$

$$F_z = \frac{y}{1+y^2z^2} + 1$$

Thus we find,

$$\frac{\partial z}{\partial x} = -\frac{1}{\frac{y}{1+y^2z^2} + 1} = \boxed{\frac{1+y^2z^2}{y+1+y^2z^2} = \frac{\partial z}{\partial x}}$$

$$\frac{\partial z}{\partial y} = \frac{-z}{\frac{y}{1+y^2z^2} + 1} = \boxed{\frac{-z}{y+1+y^2z^2} = \frac{\partial z}{\partial y}}$$

§ 11.5 #37 Consider $z = f(x, y)$ where $x = r \cos \theta$ and $y = r \sin \theta$.

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r} [f(x, y)] = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta)$$

Then calculate,

$$\begin{aligned} \left(\frac{\partial z}{\partial r} \right)^2 &= \left(\frac{\partial z}{\partial x} \right)^2 \cos^2 \theta + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cancel{\sin \theta \cos \theta} + \left(\frac{\partial z}{\partial y} \right)^2 \sin^2 \theta \\ + \left(\frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 \right) &= \left(\frac{\partial z}{\partial x} \right)^2 \sin^2 \theta - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cancel{\sin \theta \cos \theta} + \left(\frac{\partial z}{\partial y} \right)^2 \cos^2 \theta \\ \boxed{\left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2} &= \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \end{aligned}$$

§ 11.5 #45 Again suppose $z = f(x, y)$ and $x = r \cos \theta$, $y = r \sin \theta$.

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left[\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right] = \cos \theta \left(\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial r} \right) + \sin \theta \left(\frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} \right) \\ &= \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 z}{\partial y \partial x} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2} \end{aligned}$$

$$\begin{aligned} \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} &= \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(-r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y} \right) \\ &= \frac{1}{r} \left\{ -\cos \theta \frac{\partial z}{\partial x} - \left(\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) \sin \theta - \sin \theta \frac{\partial z}{\partial y} + \cos \theta \left(\frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} \right) \right\} \\ &= \underbrace{\frac{-1}{r} \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right)}_{-\frac{1}{r} \frac{\partial z}{\partial r}} + \frac{\partial^2 z}{\partial x^2} \sin^2 \theta - 2 \frac{\partial^2 z}{\partial x \partial y} \cancel{\sin \theta \cos \theta} + \frac{\partial^2 z}{\partial y^2} \cos^2 \theta \end{aligned}$$

Thus we see the offending terms cancel leaving

$$\boxed{\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = -\frac{1}{r} \frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}}$$