

§11.4 #2 Consider  $z = \exp(x^2 - y^2)$  find tangent plane to surface at  $(1, -1, 1)$ . We calculate,

$$\frac{\partial z}{\partial x} = 2x \exp(x^2 - y^2) \quad \therefore \frac{\partial z}{\partial x}\Big|_{(1,-1)} = 2e^0 = 2$$

$$\frac{\partial z}{\partial y} = -2y \exp(x^2 - y^2) \quad \therefore \frac{\partial z}{\partial y}\Big|_{(1,-1)} = -2(-1)e^0 = 2$$

Then the eq<sup>n</sup> of the tangent plane to  $z = f(x, y)$  at  $(x_0, y_0)$ ,

$$z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0, y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0, y_0)}(y - y_0)$$

In particular, for  $(x_0, y_0, z_0) = (1, -1, 1)$ ,

$$\boxed{z - 1 = 2(x - 1) + 2(y + 1)}$$

Remark: I prefer to look at  $F(x, y, z) = \exp(x^2 - y^2) - z$

then  $\nabla F \equiv \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$  gives the normal to the tangent planes of the surface  $F = 0$ .

In this problem, we've already calculated  $F_x$  and  $F_y$  at  $(1, -1, 1)$  and  $F_z = -1$  thus

$$\nabla F\Big|_{(1,-1,1)} = \langle 2, 2, -1 \rangle$$

then the plane at  $(1, -1, 1)$  is simply

$$2(x - 1) + 2(y + 1) - (z - 1) = 0$$

which is the same as we found via the text's formula.

The Point: Not all surfaces come nicely packaged as  $z = f(x, y)$ . Sometimes we can only define  $z$  implicitly.

In all cases we can rewrite the condition defining a surface in  $\mathbb{R}^3$  in terms of  $F(x, y, z) = 0$ .

Then at  $(x_0, y_0, z_0)$  on the surface  $F(x, y, z) = 0$

we can calculate the eq<sup>n</sup> of the tangent plane as follows,

$$\nabla F\Big|_{(x_0, y_0, z_0)} \bullet (x - x_0, y - y_0, z - z_0) = 0$$

Remark: one possible exception parametrically described surfaces

§11.4 #10 Find linearization of  $f(x,y) = x/y$  at  $(6,3)$ .

Notice  $\frac{\partial f}{\partial x} = \frac{1}{y}$  and  $\frac{\partial f}{\partial y} = -x/y^2$ . These are continuous at  $(6,3)$  so  $f(x,y)$  is differentiable at  $(6,3)$ .

$$\begin{aligned} L(x,y) &= f(6,3) + \left. \frac{\partial f}{\partial x} \right|_{(6,3)} (x-6) + \left. \frac{\partial f}{\partial y} \right|_{(6,3)} (y-3) \\ &= \frac{6}{3} + \frac{1}{3}(x-6) - \frac{6}{9}(y-3) \\ &= \boxed{\frac{1}{3}x - \frac{2}{3}y + 2 = L(x,y)} \end{aligned}$$

Remark: In two dimensions  $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$  then we can write  $L(x,y) = f(x_0, y_0) + [(\nabla f)(x_0, y_0)] \cdot \langle x-x_0, y-y_0 \rangle$  most everything can be written nicely using the " $\nabla$ " operator.

§11.4 #20  $u = e^{-t} \sin(s+2t)$  so  $u$  is a funct. of  $t$  and  $s$ ,

$$\begin{aligned} du &= \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt \\ &= e^{-t} \cos(s+2t) ds + [-e^{-t} \sin(s+2t) + e^{-t} \cdot 2 \cos(s+2t)] dt \\ &= \boxed{e^{-t} \cos(s+2t) ds + e^{-t} [2 \cos(s+2t) - \sin(s+2t)] dt = du} \end{aligned}$$

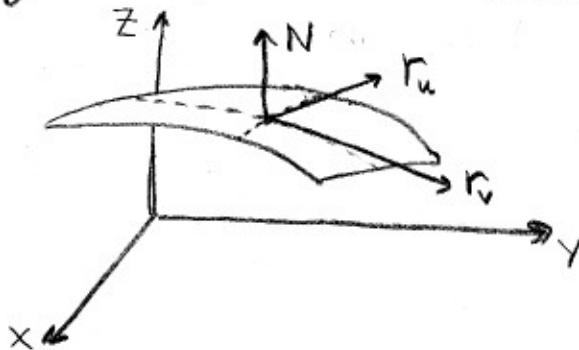
§11.4 #22  $w = xy \exp(xz)$  find the total-differential

$$\begin{aligned} dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz \\ &= [y \exp(xz) + xyz \exp(xz)] dx + [x \exp(xz)] dy + [x^2 y \exp(xz)] dz \\ &= \boxed{e^{xz} \{ (y + xyz) dx + x dy + x^2 y dz \} = dw} \end{aligned}$$

§11.4 #35 Suppose we have a parametrically defined surface,

$$\mathbf{r}(u, v) = \langle u^2, 2u \sin v, u \cos v \rangle$$

find the tangent plane at  $\mathbf{r}(1, 0) = \langle 1, 0, 1 \rangle$ . This case is different than the early problems in this section. basically the picture is something like the following



$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

Now all of this should be done at the point of interest where  $u=1$  and  $v=0$ . Calculate them,

$$\frac{\partial \mathbf{r}}{\partial u} = \langle 2u, 2\sin v, \cos v \rangle \quad \frac{\partial \mathbf{r}}{\partial u}(1, 0) = \langle 2, 0, 1 \rangle$$

$$\frac{\partial \mathbf{r}}{\partial v} = \langle 0, 2u \cos v, -u \sin v \rangle \quad \frac{\partial \mathbf{r}}{\partial v}(1, 0) = \langle 0, 2, 0 \rangle$$

$$(\mathbf{r}_u \times \mathbf{r}_v)_{(1,0)} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = -2\hat{i} + 4\hat{k} = \langle -2, 0, 4 \rangle$$

We have the normal  $\mathbf{N} = \langle -2, 0, 4 \rangle$  and a point  $(1, 0, 1)$  the tangent plane has the eqn,

$$-2(x-1) + 4(z-1) = 0$$

§ 11.4 #37

Consider the parametrized surface

$$\mathbf{r}(u, v) = \langle u, \ln(uv), v \rangle$$

$$\mathbf{r}_u = \langle 1, \frac{1}{uv} \cdot v, 0 \rangle = \langle 1, \frac{1}{u}, 0 \rangle$$

$$\mathbf{r}_v = \langle 0, \frac{1}{v}, 1 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{1}{u} & 0 \\ 0 & \frac{1}{v} & 1 \end{vmatrix} = \left\langle \frac{1}{u}, -1, \frac{1}{v} \right\rangle = \mathbf{N}(u, v)$$

We could find the tangent plane most anywhere now, but let's consider  $u=1, v=1$  where

$$\mathbf{r}(1, 1) = \langle 1, \ln(1), 1 \rangle = \langle 1, 0, 1 \rangle.$$

$$\mathbf{N}(1, 1) = \langle 1, -1, 1 \rangle$$

Hence the tangent plane is,

$$x - 1 - (y - 0) + z - 1 = 0 \Rightarrow x - y + z - 2 = 0$$

Remark: We have seen two methods of describing a surface in  $\mathbb{R}^3$ .

1.) As a level surface ;  $F(x, y, z) = 0$

2.) As a parametrized surface ;  $\mathbf{r}(u, v) = (x, y, z)$ .

The method to find the tangent plane at  $(x_0, y_0, z_0)$  on the surface followed one of the two strategies

$$1.) (\nabla F(x_0, y_0, z_0)) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$2.) \mathbf{N}(u_0, v_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

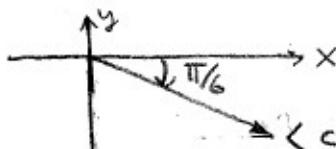
where  $\mathbf{N}(u, v) = \mathbf{r}_u \times \mathbf{r}_v$  and  $\mathbf{r}(u_0, v_0) = (x_0, y_0, z_0)$

You should be comfortable with both cases.

§11.6 #5] Let  $f(x,y) = \sqrt{5x-4y}$  find  $Df$  at  $(4,1)$  in  $\Theta = -\pi/6$  direction

$$(\nabla f)(xy) = \left\langle \frac{15}{2\sqrt{5x-4y}}, \frac{-2}{\sqrt{5x-4y}} \right\rangle \Rightarrow (\nabla f)(4,1) = \left\langle \frac{5}{2\sqrt{16}}, \frac{-2}{\sqrt{16}} \right\rangle$$

Thus  $(\nabla f)(4,1) = \left\langle \frac{5}{8}, -\frac{1}{2} \right\rangle$ . Now lets find the unit vector in the  $\Theta = -\pi/6$  direction, here  $\Theta$  is the usual polar coordinate.



$$\left\langle \cos(-\pi/6), \sin(-\pi/6) \right\rangle = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = u$$

$$\begin{aligned} D_u f(4,1) &= (\nabla f)(4,1) \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle \\ &= \left\langle \frac{5}{8}, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle \\ &= \boxed{\frac{5\sqrt{3}}{16} + \frac{1}{4}} \end{aligned}$$

Remark: we could have used  $D_u f(x,y) = f_x(x,y) \cos \Theta + f_y(x,y) \sin \Theta$ , however I'd prefer we learn to do things from basic principles as opposed to just plug & chug. On a test I'd expect you to prove the formula  $D_u f(x,y) = f_x(x,y) \cos \Theta + f_y(x,y) \sin \Theta$  if you used it. The starting point for us is  $D_u f(x,y) = (\nabla f)(xy) \cdot u$ .

§11.6 #7] Let  $f(x,y) = 5xy^2 - 4x^3y$ .

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 5y^2 - 12x^2y, 10xy - 4x^3 \right\rangle$$

$$(\nabla f)(1,2) = \left\langle 20 - 12(1)(2), 10(1)(2) - 4(1) \right\rangle = \left\langle -4, 16 \right\rangle$$

The rate of change of  $f$  at the point  $P = (1,2)$  in the  $u = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$  direction is the directional derivative, we should check that

$u = \hat{u}$  notice  $|u| = \sqrt{\frac{1}{13^2}(5^2 + 12^2)} = \sqrt{\frac{1}{169}(169)} = 1 \therefore u$  is unit vector. Thus,

$$\begin{aligned} (D_u f)(P) &= (\nabla f)(P) \cdot u \\ &= \left\langle -4, 16 \right\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle \\ &= \frac{1}{13}(-20 + 192) = \boxed{\frac{172}{13}} \end{aligned}$$

§11.6 #12 Let  $f(x, y) = \ln(x^2 + y^2)$  find  $(D_v f)(2, 1)$  for  $v = \langle -1, 2 \rangle$ .

Notice that  $|v| = \sqrt{5}$  thus  $\hat{v} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$ . You can check  $|\hat{v}| = 1$ .

$$\nabla f = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle$$

$$(\nabla f)(2, 1) = \left\langle \frac{4}{5}, \frac{2}{5} \right\rangle$$

$$(\nabla f)(2, 1) \cdot \left( \frac{1}{\sqrt{5}} \langle -1, 2 \rangle \right) = \frac{1}{\sqrt{5}} \frac{1}{5} \langle 4, 2 \rangle \cdot \langle -1, 2 \rangle = \frac{1}{5\sqrt{5}} (-4 + 4) = 0$$

Thus we find  $(D_{\hat{v}} f)(2, 1) = 0$

§11.6 #14 Let  $v = \langle 1, 2, 3 \rangle$  then  $\hat{v} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$  has  $|\hat{v}| = 1$ .

Suppose  $f(x, y, z) = x/(y+z)$ . Find  $(D_{\hat{v}} f)(4, 1, 1)$ ,

$$\nabla f = \left\langle \frac{1}{y+z}, \frac{-x}{(y+z)^2}, \frac{-x}{(y+z)^2} \right\rangle$$

$$(\nabla f)(4, 1, 1) = \left\langle \frac{1}{2}, -\frac{4}{4}, -\frac{4}{4} \right\rangle = \frac{1}{2} \langle 1, -2, -2 \rangle$$

$$(\nabla f)(4, 1, 1) \cdot \hat{v} = \frac{1}{2} \frac{1}{\sqrt{14}} \langle 1, -2, -2 \rangle \cdot \langle 1, 2, 3 \rangle = \frac{1}{2\sqrt{14}} (1 - 4 - 6) = \boxed{\frac{-9}{2\sqrt{14}}}$$

Therefore we find  $(D_{\hat{v}} f)(4, 1, 1) = \frac{-9}{2\sqrt{14}}$

Remark: If we have  $f(x, y)$  then  $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ . If we have  $f(x, y, z)$  then  $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ . If we have  $f(x_1, x_2, \dots, x_n)$  then  $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle = (\partial_i f) e_i$ .  
Assuming that  $x_1, x_2, \dots, x_n$  are CARTESIAN COORDINATES. In other coordinate systems the calculation of  $\nabla f$  is more subtle.

§11.6 #18 Consider  $f(x, y, z) = x^2 + y^2 + z^2$ . Find the directional derivative of  $f$  at  $(2, 1, 3)$  in the direction of the origin. That is the  $(-2, -1, -3)$  direction, we need a unit vector so  $\hat{u}$  by length  $\sqrt{4+1+9} = \sqrt{14}$  to construct,

$$\hat{u} = \frac{1}{\sqrt{14}}(-2, -1, -3)$$

We find the gradient of  $f$ ,

$$\nabla f = \langle 2x, 2y, 2z \rangle \Rightarrow (\nabla f)(2, 1, 3) = \langle 4, 2, 6 \rangle.$$

Hence,

$$(D_{\hat{u}} f)(2, 1, 3) = \frac{-1}{\sqrt{14}} \langle 4, 2, 6 \rangle \cdot \langle -2, -1, -3 \rangle = \frac{-1}{\sqrt{14}}(8+2+18) = \frac{-28}{\sqrt{14}}$$

$$\text{Since } \frac{\partial f}{\partial r} = \frac{\partial f}{\sqrt{x^2+y^2+z^2}} = 2\sqrt{14} \text{ we find } (D_{\hat{u}} f)(2, 1, 3) = -2\sqrt{14}$$

Remark: level surfaces of  $f(x, y, z)$  are collections of points that satisfy  $x^2 + y^2 + z^2 = k = c^2$  (we can restrict discussion to the case  $k > 0$  so  $\exists c > 0$  such that  $c^2 = k$  namely  $\sqrt{k}$ .) Anyway notice  $\nabla f = 2\langle x, y, z \rangle$ , this is twice  $\vec{r} = (x, y, z)$  it points in the radial direction. We note that  $(\nabla f)(P)$  will give the normal vector to the tangent plane of  $x^2 + y^2 + z^2 = c^2$  at the point  $P$ .

§11.6 #21 Job  $f(x, y, z) = \ln(xy^2z^3)$  find the maximum rate of change of  $f$  at the point  $(1, -2, -3)$ .

(?) apparently the text thinks  $\ln(\text{negative } \#)$  makes sense now. Notice, since  $\ln(-2)^2(-3)^3 = \ln(4)(-27) = \underline{\underline{-108}}$  this means  $f(1, -2, -3) = \ln(-108)$ . Now there is an interpretation of  $\ln(-108)$  in complex variables, but obviously this is not the point.

• Easy fix find max rate of change of  $f$  at  $(1, -2, -3)$ .

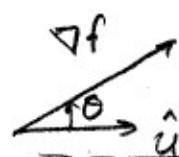
$$f(x, y, z) = \ln(x) + 2\ln(y) + 3\ln(z)$$

$$\nabla f = \langle \frac{1}{x}, \frac{2}{y}, \frac{3}{z} \rangle$$

$$(\nabla f)(1, -2, -3) = \langle 1, -1, 1 \rangle$$

$$|\nabla f(1, -2, -3)| = \sqrt{3}$$

$$(D_{\hat{u}} f)(1, -2, -3) = \langle 1, -1, 1 \rangle \cdot \hat{u} = \sqrt{3} \cos \theta$$



Thus  $\hat{u} = \frac{1}{\sqrt{3}} \langle 1, -1, 1 \rangle$  gives  $\theta = 0$  and hence the max change of  $f$  at  $(1, -2, -3)$  namely  $\sqrt{3}$

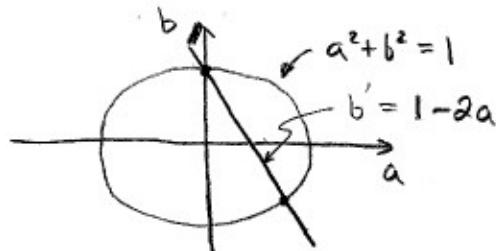
§ 11.6 #24] Find directions in which  $f(x,y) = x^2 + \sin(xy)$  has directional derivative at  $(1,0)$  with value 1. That is find  $\hat{u}$  such that  $(D_{\hat{u}} f)(1,0) = 1$ . For our convenience let us define  $a, b$  unknowns such that  $\hat{u} = \langle a, b \rangle$ . ( $a^2 + b^2 = 1$ )

$$\begin{aligned}\nabla f &= \left\langle \frac{\partial}{\partial x} [x^2 + \sin(xy)], \frac{\partial}{\partial y} [x^2 + \sin(xy)] \right\rangle \quad \text{chain-rule.} \\ &= \left\langle 2x + \cos(xy) \frac{\partial}{\partial x} [xy], \cos(xy) \frac{\partial}{\partial y} [xy] \right\rangle \\ &= \langle 2x + y \cos(xy), x \cos(xy) \rangle\end{aligned}$$

Now  $(\nabla f)(1,0) = \langle 2, 1 \rangle$ . We wish to study  $(D_{\hat{u}} f)(1,0) = 1$ , that is,

$$(\nabla f)(1,0) \cdot \langle a, b \rangle = \langle 2, 1 \rangle \cdot \langle a, b \rangle = 2a + b = 1$$

The eq<sup>n</sup>  $2a+b$  has only many sol<sup>n</sup>'s But we also demand that  $a^2+b^2=1$  since we wish to find the directions in which  $(D_{\hat{u}} f)(1,0) = 1$ .



- you can see we get two sol<sup>n</sup>'s from the graph.
  - algebraically we find them as follows,
- $$\begin{aligned}1 &= a^2 + b^2 = a^2 + (1-2a)^2 : \text{substituting} \\ &= a^2 + 1 - 4a + 4a^2 \\ &= 5a^2 - 4a + 1 \\ \therefore 5a^2 - 4a &= a(5a-4) = 0 \\ \underline{a=0} \quad \text{or} \quad \underline{a=4/5} \end{aligned}$$

Thus  $\hat{u} = \langle a, b \rangle = \langle a, 1-2a \rangle$  should be  $\boxed{\langle 0, 1 \rangle \text{ or } \langle 4/5, 3/5 \rangle}$ .

§ 11.6 #33 Properties of the gradients. Suppose  $a, b \in \mathbb{R}$  and  $u, v$  are differentiable,

$$\nabla(au+ bv) = \partial_i (au+ bv) e_i = [a(\partial_i u) + b(\partial_i v)] e_i = a(\partial_i u) e_i + b(\partial_i v) e_i$$

Therefore we find that the gradient has  $\boxed{\nabla(au+ bv) = a\nabla u + b\nabla v}$

Notice the critical step was  $\partial_i (au+ bv) = a\partial_i u + b\partial_i v$  this is just shorthand notation for the following (here  $i=1, 2, \dots, n = \dim(\text{domain}(u))$ )

$$\frac{\partial}{\partial x_i} (au+ bv) = a \frac{\partial u}{\partial x_i} + b \frac{\partial v}{\partial x_i}$$

The other properties follow similarly, we just do the product, quotient or power-rule  $n$ -times, once for each  $i$ .

§11.6 #33 continued My sol<sup>12</sup> uses the Einstein notation, the brute-force version is not much longer.

$$\begin{aligned}
 \nabla(au+bv) &= \left\langle \frac{\partial}{\partial x}(au+bv), \frac{\partial}{\partial y}(au+bv) \right\rangle \\
 &= \left\langle a \frac{\partial u}{\partial x} + b \frac{\partial v}{\partial x}, a \frac{\partial u}{\partial y} + b \frac{\partial v}{\partial y} \right\rangle \\
 &= \left\langle a \frac{\partial u}{\partial x}, a \frac{\partial u}{\partial y} \right\rangle + \left\langle b \frac{\partial v}{\partial x}, b \frac{\partial v}{\partial y} \right\rangle \\
 &= a \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + b \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle \\
 &= a \nabla u + b \nabla v.
 \end{aligned}$$

§11.6 #28 Suppose  $T(x, y, z) = 200 e^{-(x^2 - 3y^2 - 9z^2)}$

is the temperature in  $^{\circ}\text{C}$  measured at  $(x, y, z)$  in meters.

$$\begin{aligned}
 (a.) \quad \nabla T &= 200 \left\langle -2x, -6y, -18z \right\rangle e^{-(x^2 - 3y^2 - 9z^2)} \\
 &= -400 e^{-x^2 - 3y^2 - 9z^2} \left\langle x, 3y, 9z \right\rangle
 \end{aligned}$$

Notice that  $(3, -3, 3) - (2, -1, 2) = (1, -2, 1)$  is a vector that points to  $(3, -3, 3)$  from  $(2, -1, 2)$ . We need a unit vector for our purposes as usual,  $\hat{u} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle$  will do.

$$\begin{aligned}
 (D_{\hat{u}} T)(2, -1, 2) &= -\frac{400}{\sqrt{6}} e^{-4-3(1)-9(4)} \left\langle 2, -3, 18 \right\rangle \cdot \langle 1, -2, 1 \rangle \\
 &= -\frac{400}{\sqrt{6}} e^{-43} (2+6+18) = \boxed{-\frac{5200}{3e^{43}} \sqrt{6}} \left( ^{\circ}\text{C}/\text{m} \right)
 \end{aligned}$$

(b\*c) The max rate of change occurs when

$$|D_{\hat{u}} T(p)| = |\nabla T(p)| = |-400 e^{-43} \langle 2, -3, 18 \rangle| = 400 e^{-43} \sqrt{4+9+(18)^2} = \boxed{400 \sqrt{337} e^{-43}}$$

This happens when  $\hat{u} \parallel \nabla T$  that is for

$$\hat{u} = \frac{1}{\sqrt{337}} \langle 2, -3, 18 \rangle$$

§11.6 #35 Find eq's of tangent plane and normal line to the surface  $x^2 - 2y^2 + z^2 + yz = 2$  at  $(2, 1, -1)$ .

Notice  $2^2 - 2(1)^2 + (-1)^2 + (1)(-1) = 4 - 2 + 1 - 1 = 2$  so the point is indeed on the surface as claimed. This is a level surface let  $F(x, y, z) = x^2 - 2y^2 + z^2 + yz - 2$  then  $F = 0$  is the surface.

$$\nabla F = \langle 2x, -4y + z, 2z + y \rangle$$

$$(\nabla F)(2, 1, -1) = \langle 4, -4 - 1, -2 + 1 \rangle = \langle 4, -5, -1 \rangle.$$

This is the normal to the level surface at  $(2, 1, -1)$ . Hence

$$4(x-2) + 5(y-1) - (z+1) = 0 \quad \text{tangent plane to } F=0 \text{ at } (2, 1, -1).$$

We have a direction  $\langle 4, -5, -1 \rangle$  and a point  $(2, 1, -1)$  the line through this point with that direction is simply

$$r(t) = (2, 1, -1) + t \langle 4, -5, -1 \rangle$$

Remark: You can solve for  $t$  and set the eq's equal to give the "symmetric" eq's for this line. But why bother?

§11.6 #51 Consider level surfaces  $F(x, y, z) = 0$  and  $G(x, y, z) = 0$ . Suppose that  $\exists P \in \mathbb{R}^3$  with  $F(P) = 0$  and  $G(P) = 0$  and  $(\nabla F)(P), (\nabla G)(P) \neq 0$ .

Show  $(\nabla F)(P) \cdot (\nabla G)(P) = 0 \iff$  Normal lines of  $F$  &  $G$  at  $P$  are  $\perp$ .

Since  $(\nabla F)(P), (\nabla G)(P)$  point in direction of normal lines to  $F$  and  $G$  at  $P$  respectively this statement is obvious. Lines are  $\perp$  if their direction vectors are perpendicular meaning their dot-product is zero. The converse is the same, normal lines point along  $(\nabla F)(P)$  and  $(\nabla G)(P)$  so the fact that the lines' direction vectors have dot-product zero  $\Rightarrow (\nabla F)(P) \cdot (\nabla G)(P) = 0$ .

§11.6 #51b Let  $F(x, y, z) = x^2 + y^2 - z^2$  and

$G(x, y, z) = x^2 + y^2 + z^2 - r^2$  give level surfaces  $F=0$  &  $G=0$ .

$$\nabla F = \langle 2x, 2y, -2z \rangle$$

$$\nabla G = \langle 2x, 2y, 2z \rangle$$

$$\nabla F \cdot \nabla G = 4\langle x, y, -z \rangle \langle x, y, z \rangle = 4(x^2 + y^2 - z^2)$$

On a point of intersection we have  $F=0$  and  $G=0$   
 thus  $x^2 + y^2 - z^2 = 0 \therefore (\nabla F) \cdot (\nabla G) = 0$ . Although not  
 asked by text, what are the points of intersection?

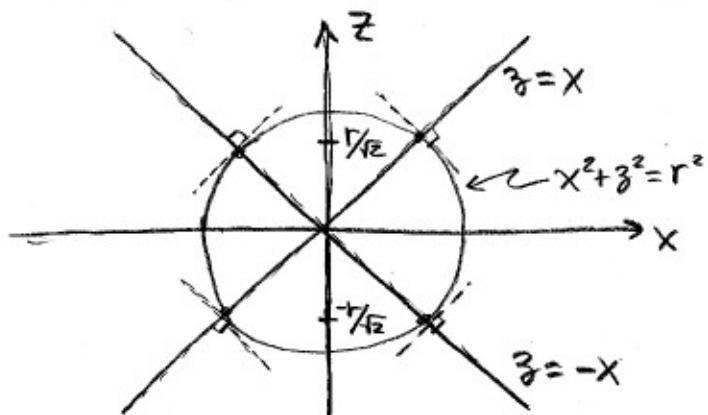
$$0 = \cancel{x^2 + y^2 + z^2} - r^2 = \cancel{x^2 + y^2} - z^2$$

$$2z^2 = r^2 \therefore z = \pm \frac{r}{\sqrt{2}} \Rightarrow x^2 + y^2 = \frac{r^2}{2}$$

there are two circles of intersection. Study  $y=0$  slice, in this plane we have simpler eq's

$$x^2 + z^2 = r^2$$

$$x^2 - z^2 = 0 \rightarrow z = \pm x$$



the three dim'l picture is obtained by rotating this about the  $z$ -axis.

I'm sure Maple would give us a nicer picture.

