

§11.7 #4 Use the level curves plotted on pg. 809 (3<sup>rd</sup> Ed. of Stewart!) to predict the location of critical points of  $f(x,y) = 3x - x^3 - 2y^2 + y^4$  and whether those critical points indicate  $f$  has a max/min or saddle point.

• Then verify graphical predictions via calculus.

• Examining the graph we see three apparent critical points.

I.)  $(-1,1)$  looks to be a local minima, the  $-2.9$  contour is smaller than the surrounding ones.

II.)  $(-1,-1)$  looks to be a critical point, besides that I'm not sure, (graphically)

III.)  $(1,0)$  looks to be a saddle point, it decreases if we go downward but upward it increases.

• Now let's produce results via calculus

$$\nabla f = \langle 3 - 3x^2, -4y + 4y^3 \rangle = \langle f_x, f_y \rangle$$

$$\nabla f = 0 \Rightarrow 3 - 3x^2 = 0 \quad \& \quad -4y + 4y^3 = 0$$

$$\Rightarrow x^2 = 1 \quad \& \quad y(y^2 - 1) = 0$$

$$\Rightarrow x = \pm 1 \quad \& \quad y = 0 \text{ or } y = \pm 1.$$

$\Rightarrow (1,0), (-1,0), (1,1), (1,-1), (-1,1), (-1,-1)$   
are critical points of  $f$

Then the 2<sup>nd</sup> partial derivatives are,

$$f_{xx} = -6x, f_{xy} = 0 = f_{yx}, f_{yy} = -4 + 12y^2$$

$$\text{We recall } D = f_{xx}f_{yy} - [f_{xy}]^2 = (-6x)(12y^2 - 4) = -72xy^2 + 24x.$$

Now use the 2<sup>nd</sup> derivative test to conclude,

Critical Point	D	$f_{xx}$	Conclusion
$(1,0)$	24	-6	local maximum at $(1,0)$ .
$(-1,0)$	-24	6	saddle point at $(-1,0)$ .
$(1,1)$	-48	-6	saddle point at $(1,1)$ .
$(1,-1)$	-48	-6	saddle point at $(1,-1)$ .
$(-1,1)$	48	6	local minimum at $(-1,1)$ .
$(-1,-1)$	48	6	local minimum at $(-1,-1)$

You can see we missed half the story graphically. Calculus is certainly safer.

§11.7 #14  $f(x,y) = (2x-x^2)(2y-y^2)$ . Find local extrema & saddle points. We use the gradient  $\nabla f = 0$  to find the critical points then apply the 2<sup>nd</sup> derivative test to determine their type (min/max/saddle).

$$\nabla f = \langle (2-2x)(2y-y^2), (2x-x^2)(2-2y) \rangle$$

This is really  $x \leftrightarrow y$  symmetric, interesting. Anyway,  $\nabla f = 0$  yields two simultaneous eq's.

$$f_x = (2-2x)(2y-y^2) = 2(1-x)y(2-y) = 0$$

$$f_y = (2x-x^2)(2-2y) = 2x(2-x)(1-y) = 0$$

$$\text{Now } f_x = 0 \Rightarrow x=1 \text{ or } y=0 \text{ or } y=2$$

$$f_y = 0 \Rightarrow x=0 \text{ or } x=2 \text{ or } y=1$$

We need at least one of both to insure  $\nabla f = \langle f_x, f_y \rangle = 0$ . This gives us the critical points,

$$(1,1), (0,0), (2,0), (0,2), (2,2)$$

Now calculate  $D = f_{xx}f_{yy} - (f_{xy})^2$ , where  $f_{xx} = -2(2y-y^2)$ ,

$$D = (-2(2y-y^2))(-2(2x-x^2)) - [(2-2x)(2-2y)]^2$$

$$D(1,1) = -2(1)(-2)(1) - 0^2 = 4.$$

$$D(0,0) = -[2(2)]^2 = -16$$

$$D(2,0) = -2(0)(-2)(4-4) - [(2-4)(2)]^2 = -16$$

$$D(0,2) = 0 - [2(2-4)]^2 = -[4]^2 = -16 \quad \text{and } D(2,2) = -16.$$

We find using the 2<sup>nd</sup> derivative test that,

critical point	D	$f_{xx}$	Conclusion
(1,1)	4	-2	local maximum
(0,0)	-16	0 (*)	saddle point
(2,0)	-16	0 (*)	saddle point
(0,2)	-16	0 (*)	saddle point
(2,2)	-16	0 (*)	saddle point

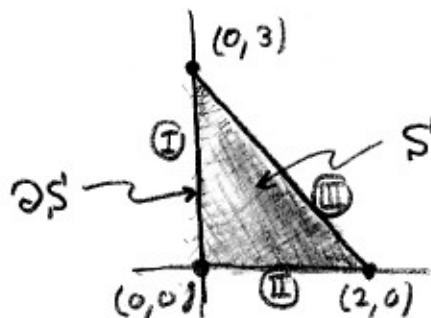
(\*) didn't need to calculate  $f_{xx}$ . The fact  $D < 0 \Rightarrow$  saddle point.

(Note polynomials are continuous so Th<sup>m</sup>(9) applies!)

H55

§ 11.7 # 25

Consider  $f(x,y) = 1 + 4x - 5y$ . Find the absolute min/max values of  $f$  on the set  $S$  below



$\partial S \equiv \text{boundary of } S$

$\partial S = \textcircled{I} \cup \textcircled{II} \cup \textcircled{III}$

$$\begin{cases} x=0 \\ 0 \leq y \leq 3 \end{cases}$$

$$\begin{cases} y=0 \\ 0 \leq x \leq 2 \end{cases}$$

$$y = -\frac{3}{2}x + 3$$
$$0 \leq x \leq 2$$

Notice that  $\nabla f = \langle 4, -5 \rangle$  thus  $f$  has no critical points. This tells us that extrema lie on the boundary.

I.  $f(x,y) = f(0,y) = 1 - 5y \equiv g(y)$

$g'(y) = -5 \Rightarrow$  extreme values on  $\partial \textcircled{I}$  namely  $y = 0$  or  $y = 3$

$$g(0) = 1 \text{ and } g(3) = -14$$

$$\underline{f(0,0) = 1} \text{ and } \underline{f(0,3) = -14}$$

II.  $f(x,y) = f(x,0) = 1 + 4x \equiv h(x)$

note  $h'(x) = 4 \neq 0$  only boundary values maximize/min h

$$h(0) = 1 \text{ while } h(2) = 9$$

$$\underline{f(0,0) = 1} \text{ and } \underline{f(2,0) = 9}$$

III.  $f(x,y) = f(x, -\frac{3}{2}x + 3) = 1 + 4x - 5(-\frac{3}{2}x + 3) \equiv f(x)$

$$\text{Now } f(x) = -14 + 4x + \frac{15}{2}x = -14 + \frac{23}{2}x$$

Thus  $f'(x) = 23/2$  thus  $f$  has no critical points, boundary points must give extrema.  $f(0) = -14 + 0 = -14$ ,  $f(2) = -14 + 23 = 9$   
hence  $\underline{f(0,3) = -14}$  and  $\underline{f(2,0) = 9}$

In each case I used max/min global extreme value theory to find the min/max values. Now in total we see by the Absolute min/max Th<sup>m</sup> on pg. 808 that

Absolute Min of  $f$  on  $S$  is  $f(0,3) = -14$

Absolute Max of  $f$  on  $S$  is  $f(2,0) = 9$

§11.7 #34 find point on plane  $x-y+z=4$  that is closest to the point  $(1, 2, 3)$ . There are two methods to solve this I.) use calculus II.) use geometry.

I.) Calculus. Define distance func from  $(1, 2, 3)$  to the plane

$$d(x, y) = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

$$= \sqrt{(x-1)^2 + (y-2)^2 + (1+y-x)^2} \quad \left. \begin{array}{l} z = 4+y-x \\ \text{on plane.} \end{array} \right]$$

Then we calculate using  $\frac{\partial}{\partial x} \sqrt{u} = \frac{1}{2\sqrt{u}} \frac{\partial u}{\partial x}$  and same for  $\frac{\partial}{\partial y}$ ,

$$\nabla d = \frac{1}{2d} \langle 2(x-1) - 2(1+y-x), 2(y-2) + 2(1+y-x) \rangle$$

$$= \frac{1}{2d} \langle 4x-4-2y, 4y-2-2x \rangle$$

$$\nabla d = 0 \Rightarrow 4x-4=2y \Rightarrow 8x-8=4y$$

$$4y-2=2x \Rightarrow 2x+2=4y \Rightarrow 6x=10$$

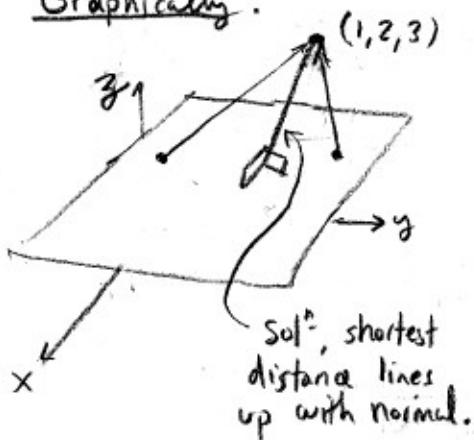
$$\therefore x = \frac{5}{3}$$

$$\frac{y = 2x-2 = 4/3}{\text{critical point is, } (\frac{5}{3}, \frac{4}{3})}$$

- Note that  $(1, 2, 3)$  is not on the plane since  $1-2+3=2 \neq 4 \therefore d \neq 0$  on the plane hence  $d$  is finite. thus  $(\frac{5}{3}, \frac{4}{3}, 4+y-x) = (\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$  is the closet point

Remark: Checking the 2nd Derivative test I leave upto you here. It is straight forward but tedious for such problems.

## II. Graphically.



I've drawn a few representative vectors from the plane to  $(1, 2, 3)$ . An arbitrary one would be

$$\mathbf{r} = (1-x, 2-y, 3-z)$$

The sol<sup>n</sup> will be  $\parallel$  to the normal.

The normal is  $\langle 1, -1, 1 \rangle$ .

§11.7 #34

Graphical Sol<sup>1</sup> (Continued), The normal line passing through  $(1, 2, 3)$  is

$$N(t) = (1, 2, 3) + t(1, -1, 1)$$

We wish to find the intersection with the plane  $x - y + z = 4$ . Let's see,

$$N(t) = \langle 1+t, 2-t, 3+t \rangle = \langle x, y, z \rangle$$

$$t = x - 1 = 2 - y = z - 3$$

$$\Rightarrow \underline{x = 3 - y} \quad \text{and} \quad \underline{z = 5 - y}$$

$$\begin{aligned} x - y + z &= 4 \Rightarrow 3 - y - y + 5 - y = 4 && \text{substituting} \\ &\Rightarrow -3y = -4 \\ &\Rightarrow \underline{y = 4/3} \Rightarrow x = \underline{5/3} \text{ and } z = \underline{11/3} \end{aligned}$$

Thus we again find that  $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$  is the closest point on plane to  $(1, 2, 3)$ . It is the point of intersection of the normal line through  $(1, 2, 3)$  and the plane.

§11.7 #49

(H58)

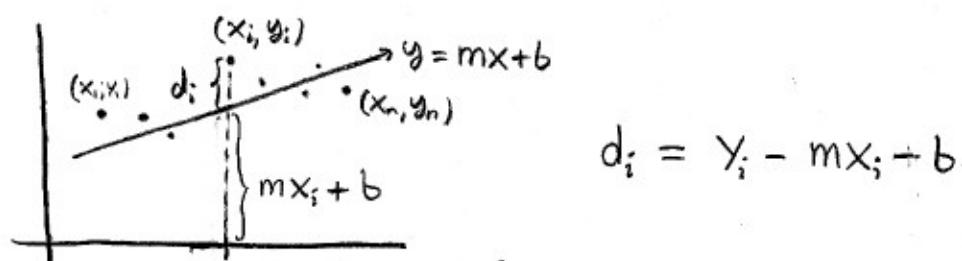
Suppose we model a relationship between  $x$  &  $y$  linearly then we expect to find  $m$  and  $b$  such that

$$y = mx + b$$

Now someone performs an experiment and collects data points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n).$$

These points are scattered about the line, but which line? We wish to find the line that fits the data best. You've probably drawn "best fit" line in lab courses to accomplish this analysis. Recall the idea is to get an equal number of points above and below the line such that the points have the same distance in total above/below the line.



Our goal is to minimize  $\sum_{i=1}^n (d_i)^2$  with respect to  $m$  and  $b$ . Define  $f(m, b) = \sum_{i=1}^n (y_i - mx_i - b)^2$ . Then calculate,

$$\frac{\partial f}{\partial m} = \sum_{i=1}^n 2(y_i - mx_i - b)(-x_i)$$

$$\frac{\partial f}{\partial b} = \sum_{i=1}^n 2(y_i - mx_i - b)(-1)$$

To have  $(\nabla f) = 0$  we need two things, (% by  $-2$  to remove  $-2$  factors),

$$\sum_{i=1}^n (x_i y_i - mx_i^2 - bx_i) = 0$$

$$\sum_{i=1}^n (y_i - mx_i - b) = 0$$

But this is the same as, since

$$m \sum_{i=1}^n x_i + bn = \sum_{i=1}^n y_i$$

$$m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

$$\sum_{i=1}^n b = b \sum_{i=1}^n 1 = b \cdot n,$$

I leave it to you to show these critical points do in fact give a minimum value for  $f(m, b)$ . (Use 2<sup>nd</sup> Der. Test)

§ 11.7 #49, Additional Comments

(H59)

We might consider the matrix form of our eq's

$$A = \begin{bmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{bmatrix} \quad d = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Then we have  $A \begin{bmatrix} m \\ b \end{bmatrix} = d$  and our sol<sup>n</sup> will be  $\begin{bmatrix} m \\ b \end{bmatrix} = A^{-1}d$  and since  $A$  is  $2 \times 2$  we have a formula for inverse

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \begin{bmatrix} \sum x_i & -n \\ -\sum x_i^2 & \sum x_i \end{bmatrix} \\ &= \frac{1}{(\sum x_i)^2 - n \sum (x_i)^2} \begin{bmatrix} \sum x_i & -n \\ -\sum x_i^2 & \sum x_i \end{bmatrix} \end{aligned}$$

Calculate then

$$\begin{aligned} \begin{bmatrix} m \\ b \end{bmatrix} &= \frac{1}{(\sum x_i)^2 - n \sum (x_i)^2} \begin{bmatrix} \sum x_i & -n \\ -\sum x_i^2 & \sum x_i \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix} \\ &= \frac{1}{(\sum x_i)^2 - n \sum (x_i)^2} \begin{bmatrix} (\sum x_i)(\sum y_i) - n \sum (x_i y_i) \\ -(\sum x_i^2) \sum (y_i) + (\sum x_i)(\sum x_i y_i) \end{bmatrix} \end{aligned}$$

Therefore the sol<sup>n</sup> is,

$$m = \frac{1}{(\sum x_i)^2 - n \sum (x_i)^2} [(\sum x_i)(\sum y_i) - n \sum (x_i y_i)]$$

$$b = \frac{1}{(\sum x_i)^2 - n \sum (x_i)^2} [-\sum (x_i)^2 (\sum y_i) + (\sum x_i)(\sum x_i y_i)]$$

I suppose the graph is easier, but hey this gives exactly the best fit line.

Remark: my presentation here is far from the slickest.

§ 11.8 #6] Let  $f(x,y) = e^{xy}$  then find min/max of  $f$  on the constraint surface  $x^3 + y^3 = 16$ . Let  $g(x,y) = x^3 + y^3$  then the method of Lagrange multipliers indicates we solve

$$\nabla f = \lambda \nabla g \text{ subject to } g = 16$$

Explicitly this yields,

$$\langle ye^{xy}, xe^{xy} \rangle = \lambda \langle 3x^2, 3y^2 \rangle \text{ with } x^3 + y^3 = 16$$

Meaning that,

$$\begin{aligned} ye^{xy} &= 3\lambda x^2 & \Rightarrow \lambda &= \frac{ye^{xy}}{3x^2} \\ xe^{xy} &= 3\lambda y^2 & \Rightarrow \lambda &= \frac{xe^{xy}}{3y^2} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{assuming } x, y \neq 0.$$

Can we assume  $x, y \neq 0$ ? Well if  $x = 0$  then  $0 = 3\lambda y^2 \Rightarrow y = 0$  but then  $x^3 + y^3 = 0 \neq 16$ . Likewise  $y = 0 \Rightarrow x = 0 \Rightarrow x^3 + y^3 \neq 16$ .

$$\lambda = \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2} \Rightarrow \frac{y}{x^2} = \frac{x}{y^2} \Rightarrow y^3 = x^3 \Rightarrow \underline{\underline{x = y}}$$

Then  $x^3 + y^3 = 2x^3 = 16 \Rightarrow x^3 = 8 \therefore \underline{\underline{x=2}}$  this shows that  $f(2,2) = e^4$  is an extreme value of  $f$ . Notice that  $x^3 + y^3 = 16$  allows for  $x < 0$  or  $y < 0$  this will cause  $\exp(xy) \rightarrow 0$  (but not equal of course). So

$e^4 = f(2,2)$  is the global maximum value and there is no global minimum value (although  $f(x,y)$  approaches zero asymptotically)

§11.8 #9  $f(x, y, z) = xyz$ , max/min subject to  $x^2 + 2y^2 + 3z^2 = 6$ .  
 that is max/min-imize w.r.t.  $g(x, y, z) = x^2 + 2y^2 + 3z^2 = 6$ .

$$\nabla f = \langle yz, xz, xy \rangle$$

$$\nabla g = \langle 2x, 4y, 6z \rangle$$

$$\begin{array}{l} \nabla f = 2\nabla g \\ \hline \hline \end{array} \quad \begin{array}{l} yz = 2x \\ xz = 4y \\ xy = 6z \end{array} \quad \begin{array}{l} xyz = 2x^2 \\ xyz = 4y^2 \\ xyz = 6z^2 \end{array}$$

Notice  $x, y, z \neq 0$  since any one of them zero  $\Rightarrow xyz = 0$   
 which in turn implies the other two are zero by eq's above.  
 Thus we can divide by  $x, y, z, \dots$

$$\frac{xyz}{2} = 2x^2 = 4y^2 = 6z^2$$

$$\Rightarrow x^2 = 2y^2 = 3z^2$$

$$\Rightarrow x^2 + 2y^2 + 3z^2 = 3x^2 = 6 \quad \therefore x^2 = 2$$

$$\therefore x = \pm\sqrt{2}$$

Now  $y^2 = \frac{1}{2}x^2 = 1$  and  $z^2 = \frac{1}{3}x^2 = \frac{2}{3}$  thus in total  
 $x = \pm\sqrt{2}$ ,  $y = \pm 1$ ,  $z = \pm\sqrt{\frac{2}{3}}$ . The possible extreme  
 values will be  $f(\pm\sqrt{2}, \pm 1, \pm\sqrt{\frac{2}{3}})$  by the method of  
 Lagrange multipliers. Let's figure out which is which,

- I)  $f(\sqrt{2}, 1, \sqrt{\frac{2}{3}}) = 2\sqrt{\frac{2}{3}}$
- II)  $f(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}) = -2\sqrt{\frac{2}{3}}$
- III)  $f(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}) = 2\sqrt{\frac{2}{3}}$
- IV)  $f(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}) = -2\sqrt{\frac{2}{3}}$
- V)  $f(\sqrt{2}, -1, \sqrt{\frac{2}{3}}) = -2\sqrt{\frac{2}{3}}$
- VI)  $f(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}) = 2\sqrt{\frac{2}{3}}$

- Maximum of  $2\sqrt{\frac{2}{3}}$  reached at cases I., III.) and VI.)
- Min. of  $-2\sqrt{\frac{2}{3}}$  reached at cases II, IV and V.

§11.8 #26 Use Lagrange multipliers to solve #34 from §11.7. That is find the point on  $x-y+z=4$  that is closest to  $(1,2,3)$ .

Minimize

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$$

Subject to  $g(x, y, z) = x-y+z=4$ . Consider then

$$\nabla f = \lambda g \quad \begin{cases} \frac{\partial f}{\partial x} = \lambda \\ \frac{\partial f}{\partial y} = -\lambda \\ \frac{\partial f}{\partial z} = \lambda \end{cases}$$

$$\therefore \frac{\lambda}{2} = x-1 = 2-y = z-3$$

$$x = 3+y$$

$$z = 5-y$$

$$\text{Then } x-y+z = 3+y-y+5-y = -3y+8 = 4 \quad \therefore y = 4/3.$$

$$\text{thus } z = 5 - 4/3 = \frac{15-4}{3} = 11/3 \quad \text{and} \quad x = 3 - 4/3 = \frac{9-4}{3} = 5/3$$

hence  $(5/3, 4/3, 11/3)$  is closest point on  $x-y+z=4$  to the point  $(1,2,3)$ .

§1a.a#4 Calculate the iterated integral

$$\begin{aligned} \int_2^4 \int_{-1}^1 (x^2+y^2) dy dx &= \int_2^4 \left( yx^2 + \frac{1}{3}y^3 \right) \Big|_{-1=y}^{1=y} dx \\ &= \int_2^4 \left[ \left( x^2 + \frac{1}{3} \right) - \left( -x^2 + \frac{1}{3}(-1) \right) \right] dx \\ &= \int_2^4 \left[ 2x^2 + \frac{2}{3} \right] dx \\ &= \frac{2}{3} \left( x^3 + x \right) \Big|_2^4 \\ &= \frac{2}{3} \left\{ (64+4) - (8+2) \right\} \\ &= \frac{2(58)}{3} = \boxed{\frac{116}{3}} \end{aligned}$$