

§12.2 #6 Calculate the double integral,

$$\begin{aligned}
 \int_1^4 \int_0^2 (x + \sqrt{y}) dx dy &= \int_1^4 \left\{ \frac{1}{2}x^2 \Big|_0^2 + x\sqrt{y} \Big|_0^2 \right\} dy \\
 &= \int_1^4 \{ 2 + 2\sqrt{y} \} dy \\
 &= \left(2y + \frac{4}{3}y^{3/2} \Big|_1^4 \right) \\
 &= \left[2(4) + \frac{4}{3}(\sqrt{4})^3 \right] - \left[2 + \frac{4}{3} \right] \\
 &= \left[8 + \frac{32}{3} \right] - \left[\frac{10}{3} \right] = \frac{24+32-10}{3} = \boxed{\frac{46}{3}}
 \end{aligned}$$

§12.2 #10 Integrate.

$$\begin{aligned}
 \int_1^2 \int_0^1 \frac{1}{(x+y)^2} dx dy &= \int_1^2 \left. \frac{-1}{(x+y)} \right|_0^1 dy \\
 &= \int_1^2 \left[\frac{-1}{(y+1)} + \frac{1}{y} \right] dy \\
 &= \left[-\ln|y+1| + \ln|y| \right]_1^2 \\
 &= [-\ln(3) + \ln(2)] - [-\ln(2) + \ln(1)] \\
 &= -\ln(3) + 2\ln(2) \\
 &= \ln(4) - \ln(3) = \boxed{\ln(4/3)}
 \end{aligned}$$

§12.2 #12 Integrate.

$$\begin{aligned}
 \int_0^1 \int_0^1 xy \sqrt{x^2+y^2} dy dx &= \int_0^1 \left\{ \int_{x^2}^{x^2+1} \sqrt{u} \cdot \frac{x du}{2} \right\} dx \quad \text{u-substitution} \\
 &= \int_0^1 x \left(\frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_{x^2}^{x^2+1} \right) dx \\
 &\qquad\qquad\qquad \left. \begin{array}{l} u = x^2 + y^2, \text{ x-fixed.} \\ y=0 \Rightarrow u=x^2 \\ y=1 \Rightarrow u=x^2+1 \\ du = 2y dy \\ \frac{1}{2}x du = xy dy \end{array} \right\}
 \end{aligned}$$

Remark: You may find it easier to go off to the side and calculate difficult integrals indefinitely. Otherwise need to change bounds as I have here.

$$\begin{aligned}
 &= \int_0^1 x \left[\frac{1}{3} [x^2+1]^{3/2} - \frac{1}{3} [x^2]^{3/2} \right] dx \\
 &= \int_0^1 \left(\frac{1}{3} x (x^2+1)^{3/2} - \frac{1}{3} x^4 \right) dx \\
 &= \left. \frac{1}{3} \left[\frac{1}{5} (x^2+1)^{5/2} - \frac{1}{5} x^5 \right] \right|_0^1 \\
 &= \frac{1}{3} \left[\left(\frac{1}{5} (\sqrt{2})^5 - \frac{1}{5} \right) - \left(\frac{1}{5} \right) \right] = \boxed{\frac{1}{15} (4\sqrt{2} - 2)}
 \end{aligned}$$

$$\begin{aligned}
 W &= x^2 - 1 \\
 dW &= 2x dx
 \end{aligned}$$

§12.2 #14 Let $R = \{(x, y) / 0 \leq x \leq \pi, 0 \leq y \leq \pi/2\}$, calculate

$$\begin{aligned}
 \iint_R \cos(x+2y) dA &= \int_0^{\pi} \left(\int_0^{\pi/2} \cos(x+2y) dy \right) dx \\
 &= \int_0^{\pi} \left(\frac{1}{2} \sin(x+2y) \Big|_{0=y}^{\pi/2=y} \right) dx \\
 &= \int_0^{\pi} \frac{1}{2} (\sin(x+\pi) - \sin(x)) dx \\
 &= \frac{1}{2} (-\cos(x+\pi) + \cos(x)) \Big|_0^{\pi} \\
 &= \frac{1}{2} (-\cos(2\pi) + \cos(\pi)) - \frac{1}{2} (-\cos(\pi) + \cos(0)) \\
 &= \frac{1}{2} (-1 - 1) - \frac{1}{2} (1 + 1) = \boxed{-2}
 \end{aligned}$$

§12.2 #16 $R = \{(x, y) / 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$$\begin{aligned}
 \iint_R \frac{1+x^2}{1+y^2} dA &= \int_0^1 \int_0^1 \frac{1+x^2}{1+y^2} dy dx \\
 &= \int_0^1 (1+x^2) dx \int_0^1 \frac{1}{1+y^2} dy \\
 &= \left(x + \frac{1}{3} x^3 \right) \Big|_0^1 \left(\tan^{-1}(y) \right) \Big|_0^1 \\
 &= \left(\frac{4}{3} \right) \left[\tan^{-1}(1) - \tan^{-1}(0) \right] = \boxed{\frac{\pi}{3}} \quad \left(\text{used } \tan(0) = 0 \Rightarrow \tan^{-1}(0) = 0 \right. \\
 &\quad \left. \tan(\pi/4) = 1 \Rightarrow \tan^{-1}(1) = \frac{\pi}{4} \right)
 \end{aligned}$$

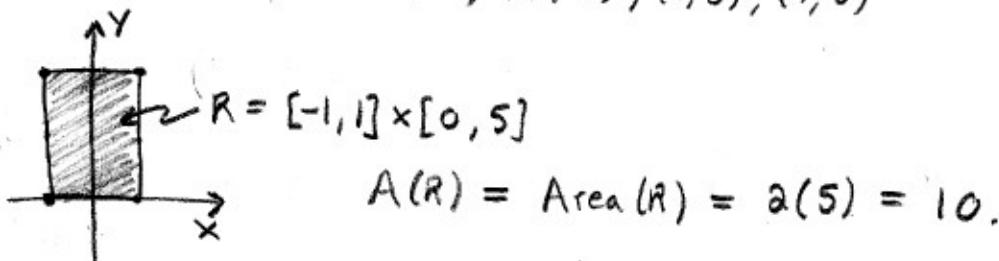
§12.2 #23 Find volume of solid bounded by $Z = 1 - x^2/4 - y^2/9$ (the top)

and $R = [-1, 1] \times [-2, 2]$ (the bottom) (the sides are $x=-1, x=1, y=2, y=-2$)

we view R as a subset of (xy) -plane. In this special case the volume is found by integrating $Z = 1 - x^2/4 + y^2/9$,

$$\begin{aligned}
 V &= \iint_R (1 - \frac{1}{4}x^2 - \frac{1}{9}y^2) dA = \int_{-1}^1 \int_{-2}^2 (1 - \frac{1}{4}x^2 - \frac{1}{9}y^2) dy dx \\
 &= \int_{-1}^1 \left[y - \frac{1}{4}x^2y - \frac{1}{27}y^3 \right]_{-2}^2 dx \\
 &= \int_{-1}^1 [4 - x^2 - 16/27] dx \quad \curvearrowright 4 - 16/27 = \frac{108-16}{27} = \frac{92}{27} \\
 &= 2 \left(\frac{92}{27} \right) - \frac{1}{3}x^3 \Big|_{-1}^1 \\
 &= 2 \left(\frac{92}{27} \right) - \frac{1}{3}(1 - (-1)) = \frac{184 - 18}{27} = \boxed{\frac{166}{27}}
 \end{aligned}$$

§12.2 #31 The average of $f(x,y) = x^2y$ over some region R is defined to be the $\iint f(x,y)dA$ divided by the area of $R = \iint_R dA = A(R)$. Let R be region with vertices $(-1,0), (-1,5), (1,5), (1,0)$



$$f_{\text{avg}} = \frac{1}{10} \int_0^5 \int_{-1}^1 x^2y \, dx \, dy = \frac{1}{10} \int_0^5 \left[\frac{1}{3}x^3y \right]_{-1}^1 \, dy = \frac{1}{10} \int_0^5 \frac{2}{3}y \, dy = \frac{2}{30} \left[\frac{y^2}{2} \right]_0^5$$

$$\therefore f_{\text{avg}} = \frac{25}{30} = \frac{5}{6}$$

Remark: So §12.2 integrals aren't any more difficult than the integrals we saw in Calc I or II. The new features really start in the next section. There we will find graphing a needed ally.

§12.3 #3 Integrate,

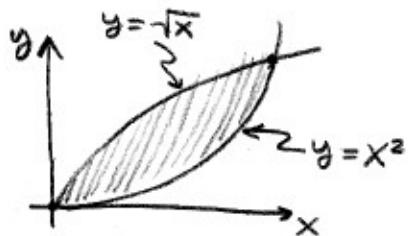
$$\begin{aligned} \int_0^1 \int_y^{e^y} \sqrt{x} \, dx \, dy &= \int_0^1 \left(\frac{2}{3}x^{3/2} \Big|_y^{e^y} \right) dy \\ &= \frac{2}{3} \int_0^1 (e^{3y/2} - y^{3/2}) dy \\ &= \frac{2}{3} \left[\frac{2}{3}e^{3y/2} - \frac{2}{5}y^{5/2} \Big|_0^1 \right] \\ &= \frac{4}{3} \left[\left(\frac{1}{3}e^{3/2} - \frac{1}{5} \right) - \left(\frac{2}{3} - 0 \right) \right] \quad \frac{1}{5} + \frac{2}{3} = \frac{3+10}{15} = \frac{13}{15} \\ &= \frac{4}{3} \left(\frac{1}{3}e^{3/2} - \frac{13}{15} \right) = \boxed{\frac{4}{9}e^{3/2} - \frac{13}{45}} \end{aligned}$$

§12.3 #10

$$\int_0^1 \int_0^y e^{y^2} \, dx \, dy = \int_0^1 (xe^{y^2} \Big|_0^y) dy = \int_0^1 ye^{y^2} dy = \frac{1}{2}e^{y^2} \Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$$

- Notice that if we had tried to integrate with respect to y first we would have been stuck since $\int e^{y^2} dy$ is not an elementary integral. Sometimes reversing the order of integration makes the problem easier.

§12.3 #12 Let $D = f(x, y) / \text{ bounded by } y = \sqrt{x} \text{ and } y = x^2 \}$



points of intersection have

$$\sqrt{x} = x^2$$

$$x = x^4$$

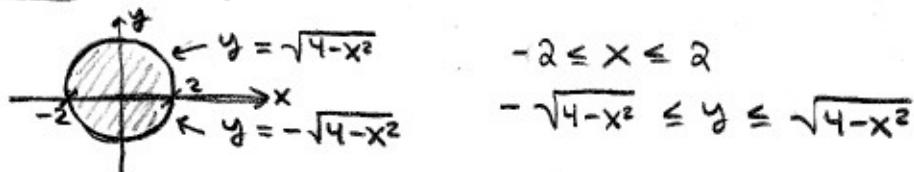
$$x(x^3 - 1) = 0 \Rightarrow x=0 \text{ or } x=1$$

Thus the region $D = \{(x, y) | 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$. Now we can do the integration, we must do it in the order below

$$\begin{aligned} \iint_D (x+y) dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y) dy dx \\ &= \int_0^1 \left(xy + \frac{1}{2}y^2 \right) \Big|_{x^2}^{\sqrt{x}} dx \\ &= \int_0^1 \left[\left(x^{3/2} + \frac{1}{2}x \right) - \left(x^3 + \frac{1}{2}x^4 \right) \right] dx \\ &= \left(\frac{2}{5}x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5 \right) \Big|_0^1 \\ &= \frac{2}{5} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{10} = \frac{4}{10} - \frac{1}{10} = \boxed{\frac{3}{10}} \end{aligned}$$

unless we do some extra work to convert to the complementary inequalities
 $0 \leq y \leq 1$
 $y^2 \leq x \leq \sqrt{y}$

§12.3 #15 $D = \{(x, y) | x^2 + y^2 \leq 4\}$

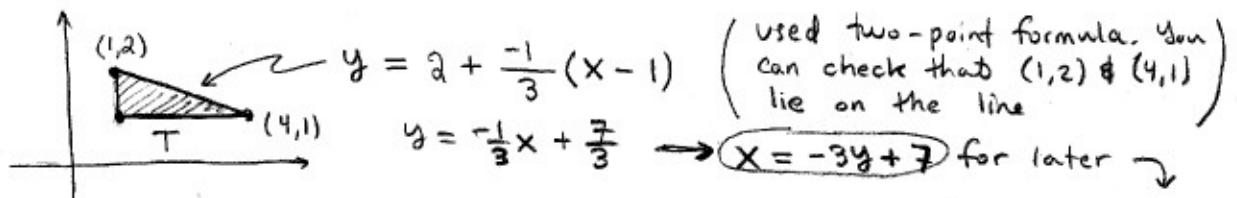


Now we can integrate,

$$\begin{aligned} \iint_D (2x-y) dA &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y) dy dx \\ &= \int_{-2}^2 \left[2xy - \frac{1}{2}y^2 \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^2 \left[4x\sqrt{4-x^2} - \frac{1}{2}(4-x^2) + \frac{1}{2}(4-x^2) \right] dx \\ &= -\frac{4}{3}(4-x^2)^{3/2} \Big|_{-2}^2 \\ &= -\frac{4}{3}(0-0) = \boxed{0} \end{aligned}$$

could have argued this is zero since $f(-x) = -f(x)$
 $\Rightarrow \int_a^a f(x) dx = 0$
 (odd fnct. trick)

§12.3 #19 find volume under $z = xy$ and above the triangle with vertices $(1,1)$, $(4,1)$ and $(1,2)$.



The triangle $T = \{(x,y) \mid 1 \leq x \leq 4, 1 \leq y \leq \frac{1}{3}(7-x)\}$

$$V = \int_1^4 \int_1^{\frac{1}{3}(7-x)} xy \, dy \, dx$$

$$= \int_1^4 \left(\frac{1}{2}xy^2 \Big|_{\frac{1}{3}(7-x)} \right) dx$$

$$= \int_1^4 \left[\frac{1}{2}x\left(\frac{1}{3}(7-x)\right)^2 - \frac{1}{2}x \right] dx$$

this is somewhat messy, I'll let you finish it. Instead I'll go another way. Notice that we can also write T as

$$T = \{(x,y) \mid 1 \leq y \leq 2, 1 \leq x \leq 7-3y\}$$

then we can integrate x first then y .

$$V = \int_1^2 \int_1^{7-3y} xy \, dx \, dy$$

$$= \int_1^2 \frac{1}{2}y x^2 \Big|_1^{7-3y} \, dy$$

$$= \int_1^2 \frac{1}{2}y((7-3y)^2 - 1) \, dy$$

$$= \int_1^2 \frac{1}{2}y(48 - 42y + 9y^2) \, dy$$

$$= \frac{1}{2} \int_1^2 (48y - 42y^2 + 9y^3) \, dy$$

$$= \frac{1}{2} \left[24y^2 - \frac{42}{3}y^3 + \frac{9}{4}y^4 \right]_1^2$$

$$= \frac{1}{2} [24(4-1) - 14(8-1) + \frac{9}{4}(16-1)]$$

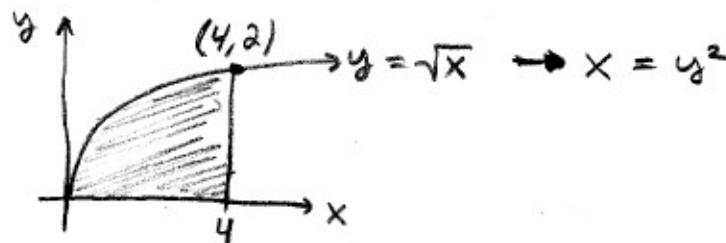
$$= \frac{1}{2} [72 - 98 + \frac{135}{4}] = \frac{1}{2} [-26 + \frac{135}{4}] = \frac{1}{2} \left[\frac{-104 + 135}{4} \right] = \boxed{\frac{31}{8}}$$

- Ok, its messy anyway you slice it.

§12.3 #33 Consider the following integral,

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y) dy dx$$

this indicates the integral is over $0 \leq y \leq \sqrt{x}$ and $0 \leq x \leq 4$.

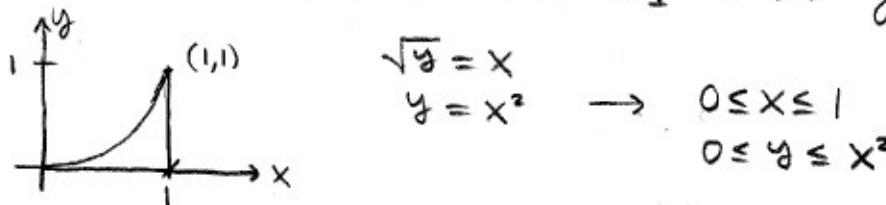


Equivalently we could say $y^2 \leq x \leq 4$ and $0 \leq y \leq 2$

$$\int_0^2 \int_{y^2}^4 f(x,y) dx dy = \int_0^4 \int_0^{\sqrt{x}} f(x,y) dy dx$$

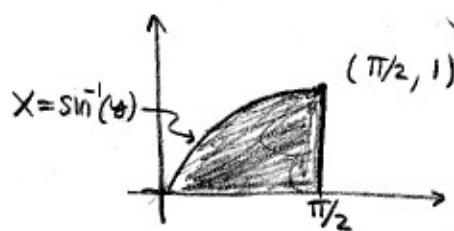
for these problems you just have to draw the picture and sort it out.

§12.3 #40 Notice that $0 \leq y \leq 1$, $\sqrt{y} \leq x \leq 1$ is the graph below



$$\begin{aligned}
 \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1} dx dy &= \int_0^1 \int_0^{x^2} \sqrt{x^3+1} dy dx \\
 &= \int_0^1 y \sqrt{x^3+1} \Big|_0^{x^2} dx \\
 &= \int_0^1 x^2 \sqrt{x^3+1} dx \\
 &= \frac{2}{9} (x^3+1)^{3/2} \Big|_0^1 \\
 &= \frac{2}{9} [(2)^{3/2} - 1] = \boxed{\frac{2}{9} [2\sqrt{2} - 1]}
 \end{aligned}$$

§12.3 #43 Consider the region $0 \leq y \leq 1$ and $\sin^{-1}(y) \leq x \leq \pi/2$



$$\begin{aligned} \sin^{-1}(y) &= x \\ \Rightarrow y &= \sin(x) \\ \Rightarrow \boxed{0 \leq x \leq \pi/2} \\ \boxed{0 \leq y \leq \sin(x)} \end{aligned}$$

Then

$$\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} \cos(x) \sqrt{1+\cos^2 x} dx dy = \int_0^{\pi/2} \int_0^{\sin(x)} \cos(x) \sqrt{1+\cos^2 x} dy dx =$$

$$\begin{aligned} &= \int_0^{\pi/2} \sin(x) \cos(x) \sqrt{1+\cos^2 x} dx \\ &= -\frac{1}{2} \frac{2}{3} (1+\cos^2 x)^{3/2} \Big|_0^{\pi/2} \\ &= -\frac{1}{3} (1 - 2^{3/2}) \\ &= \boxed{\frac{1}{3}(2\sqrt{2}-1)} \end{aligned}$$

$$\begin{cases} u = 1 + \cos^2 x \\ du = -2\cos x \sin x dx \end{cases}$$

§12.7 #2 evaluate the integral three different ways. The region of integration is $E = \{(x, y, z) \mid -1 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1\}$

$$\begin{aligned} \iiint_E (xz - y^3) dV &= \int_{-1}^1 \int_0^2 \int_0^1 (xz - y^3) dz dy dx \\ &= \int_{-1}^1 \int_0^2 \left(\frac{1}{2}x - y^3\right) dy dx \\ &= \int_{-1}^1 \left(x - \frac{1}{4}(16)\right) dx = -4(2) = \boxed{-8} \end{aligned}$$

$$\begin{aligned} \iiint_E (xz - y^3) dV &= \int_0^1 \int_0^2 \int_{-1}^1 (xz - y^3) dx dy dz \\ &= \int_0^1 \int_0^2 (-xy^3) \Big|_{-1}^1 dy dz \\ &= \int_0^1 \int_0^2 -2y^3 dy dz \\ &= \int_0^1 -\frac{1}{4}(2)^4 dz = -8z \Big|_0^1 = \boxed{-8} \end{aligned}$$

here xz is an odd-function integrated over symmetric interval about zero \Rightarrow vanishes.

There are four other ways to iterate the integral. Each will yield -8. This is Fubini's Th for \iiint in action.

§12.7 #5

$$\begin{aligned}
 \int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} 3e^y dx dz dy &= \int_0^3 \int_0^1 3e^y x \Big|_0^{\sqrt{1-z^2}} dz dy \\
 &= \int_0^3 \int_0^1 3\sqrt{1-z^2} e^y dz dy \\
 &= \int_0^3 \left(-\frac{1}{3}(1-z^2)^{3/2} e^y \Big|_0^1 \right) dy \\
 &= \int_0^3 \frac{1}{3} e^y dy = \boxed{\frac{1}{3}(e^3 - 1)}
 \end{aligned}$$

§12.7 #6

$$\begin{aligned}
 \int_0^1 \int_0^z \int_0^y 3e^{-y^2} dx dy dz &= \int_0^1 \int_0^z \left(3e^{-y^2} x \Big|_0^y \right) dy dz \\
 &= \int_0^1 \int_0^z 3e^{-y^2} y dy dz \\
 &= \int_0^1 \left(-\frac{1}{2} 3e^{-y^2} \Big|_{0=y}^{y=z} \right) dz \\
 &= \int_0^1 \left(-\frac{1}{2} 3e^{-z^2} + \frac{1}{2} 3 \right) dz \\
 &= \left(\frac{1}{4} e^{-z^2} + \frac{1}{4} z^2 \Big|_0^1 \right) \\
 &= \frac{1}{4} (e^{-1} + 1 - 1) = \boxed{\frac{1}{4e}}
 \end{aligned}$$

§12.7 #8

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, x \leq z \leq 2x\}$$

We must integrate dz then dy then dx . That is the natural order here.

$$\begin{aligned}
 \iiint_E yz \cos(x^5) dV &= \int_0^1 \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx \\
 &= \int_0^1 \cos(x^5) \left(\int_0^x \int_x^{2x} yz dz dy \right) dx \\
 &= \int_0^1 \cos(x^5) \left(\int_0^x \frac{1}{2} y [(2x)^2 - x^2] dy \right) dx \\
 &= \int_0^1 \frac{3}{2} x^2 \cos(x^5) \left(\int_0^x y dy \right) dx \\
 &= \int_0^1 \frac{3}{4} x^4 \cos(x^5) dx = \frac{3}{20} \sin(x^5) \Big|_0^1 = \boxed{\frac{3}{20} \sin(1)}
 \end{aligned}$$

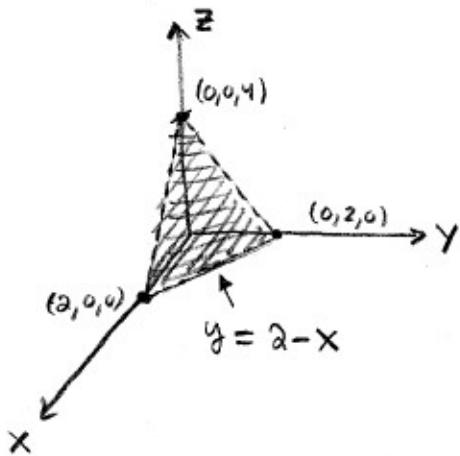
§12.7 #10 Let $E \subset \mathbb{R}^3$ bounded by $x=0, y=0, z=0$ and $2x+2y+z = 4$
 Notice that this plane passes through the first octant.

$$(xz)\text{-plane } (y=0) : 2x+z=4 \therefore z=4-2x$$

$$(xy)\text{-plane } (z=0) : 2x+2y=4 \therefore y=2-x$$

$$(yz)\text{-plane } (x=0) : 2y+z=4 \therefore z=4-2y$$

These details are not strictly speaking necessary but sometime it helps to get some additional details to help insure graph is correct.



So we can describe the region of integration as

$$0 \leq z \leq 4-2x-2y$$

but what about x & y ? Note on (xy) -plane we have

$$0 \leq y \leq 2-x$$

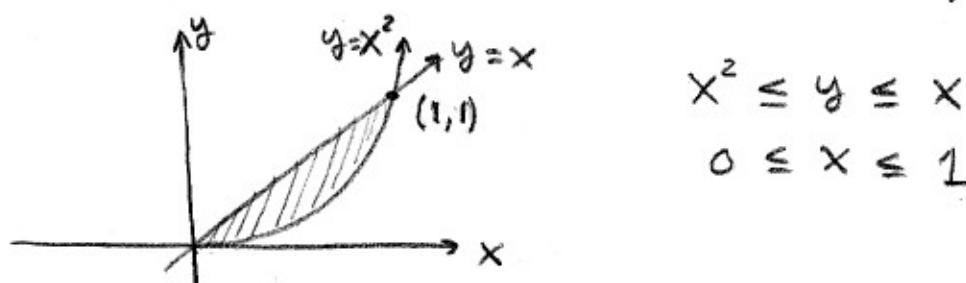
And finally

$$0 \leq x \leq 2$$

Now integrate,

$$\begin{aligned} \iiint_E y \, dV &= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y \, dz \, dy \, dx \\ &= \int_0^2 \int_0^{2-x} y(4-2x-2y) \, dy \, dx \\ &= \int_0^2 \left[y^2(2-x) - \frac{2}{3}y^3 \right]_0^{2-x} \, dx \\ &= \int_0^2 \left[(2-x)^3 - \frac{2}{3}(2-x)^3 \right] \, dx \\ &= \int_0^2 \frac{1}{3}(2-x)^3 \, dx \\ &= \frac{1}{12}(2-x)^4 \Big|_0^2 \\ &= -\frac{1}{12}(0-16) = \frac{16}{12} = \boxed{\frac{4}{3}} \end{aligned}$$

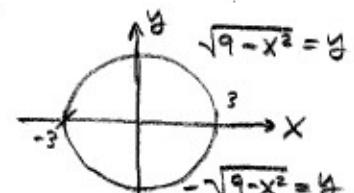
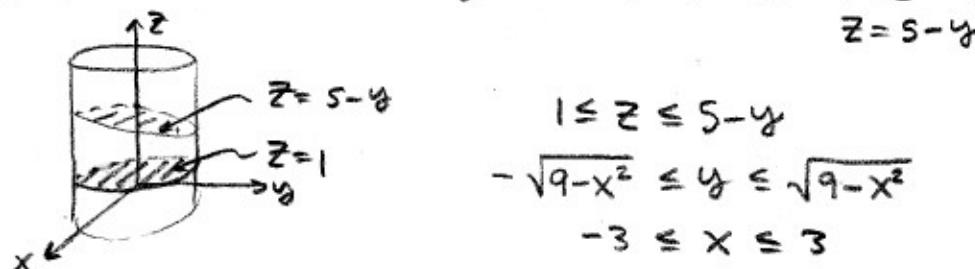
§12.7 #14 $E \subset \mathbb{R}^3$ bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$ and $z = 0$. We can bound z to begin, $0 \leq z \leq x$. Then a 2-dim'l picture will do,



Now integrate

$$\begin{aligned} \iiint_E (x+2y) dV &= \int_0^1 \int_{x^2}^x \int_0^x (x+2y) dz dy dx \\ &= \int_0^1 \int_{x^2}^x (x^2 + 2yx) dy dx \\ &= \int_0^1 (x^2 y + xy^2) \Big|_{x^2}^x dx \\ &= \int_0^1 [x^3 + x^3 - x^4 - x^5] dx \\ &= \left(\frac{2}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6 \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{5} - \frac{1}{6} = \frac{15-6-5}{30} = \boxed{\frac{2}{15}} \end{aligned}$$

§12.7 #19 Find volume enclosed by $x^2 + y^2 = 9$ and $y+z=5$ and $z=1$



$$\begin{aligned} V &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y^2} dz dy dx = \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-y^2) dy dx \quad \text{(odd fn't about even interval)} \\ &= \int_{-3}^3 8\sqrt{9-x^2} dx \\ &= \int_{-\pi/2}^{\pi/2} 24 \cdot 3 \cos^2 \theta d\theta \quad \left\{ \begin{array}{l} x = 3 \sin \theta \\ 9-x^2 = 9 \cos^2 \theta \\ dx = 3 \cos \theta d\theta \end{array} \right. \\ &= 72 \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\theta) d\theta = \boxed{36\pi} \end{aligned}$$

§12.7 #31 Consider $\int_0^1 \int_{\sqrt{x}}^{1-y} \int_0^{1-y} f(x, y, z) dz dy dx$. Find five other orders of iterating this integral.

$$0 \leq z \leq 1-y$$

$$\sqrt{x} \leq y \leq 1$$

$$0 \leq x \leq 1$$

$$0 \leq z \leq 1-y$$

OR

$$0 \leq x \leq y^2$$

$$0 \leq y \leq 1$$

OROR

$$0 \leq x \leq y^2$$

$$0 \leq x \leq y^2$$

$$0 \leq y \leq 1-z$$

$$0 \leq z \leq 1-y$$

$$0 \leq z \leq 1$$

$$0 \leq y \leq 1$$

OROR

$$\sqrt{x} \leq y \leq 1-z$$

$$\sqrt{x} \leq y \leq 1-z$$

$$0 \leq z \leq 1-\sqrt{x}$$

$$0 \leq x \leq (1-z)^2$$

$$0 \leq x \leq 1$$

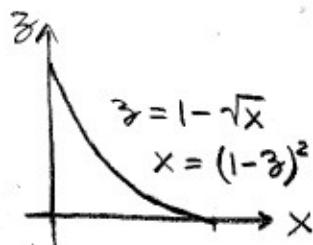
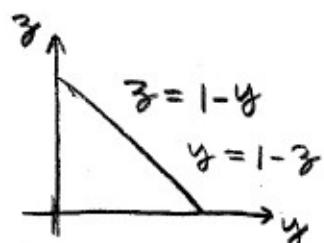
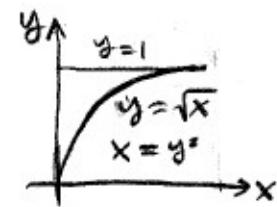
$$0 \leq z \leq 1$$

These are equivalent views of the volume being integrated over. Thus

$$\int_0^1 \int_{\sqrt{x}}^{1-y} \int_0^{1-y} f dz dy dx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f dz dx dy$$

$$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f dx dy dz = \int_0^1 \int_0^{1-y} \int_0^{y^2} f dx dz dy$$

$$= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f dy dz dx = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f dy dx dz$$



Volume E's projections onto
coordinates planes