

§10.5 #13 Consider $r(u,v) = \langle u \cos v, u \sin v, v \rangle$ try to match with graph on p. 733 of 3rd ed. of Stewart.

(1) fix v and let u vary,

$$r(u, v_0) = \langle u \cos v_0, u \sin v_0, v_0 \rangle \equiv \alpha(u)$$

this is a line in the $z = v_0$ plane in the $\langle \cos v_0, \sin v_0, 0 \rangle$ direction

(2) fix $u = u_0$ let v vary,

$$r(u_0, v) = \langle u_0 \cos v, u_0 \sin v, v \rangle \equiv \beta(v)$$

this is a helix of radius u_0 and slope one

Notice then that these u, v coordinate curves are perpendicular. This can be seen by checking their tangents,

$$\alpha'(u) = \langle \cos v_0, \sin v_0, 0 \rangle$$

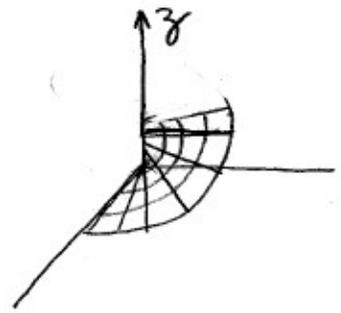
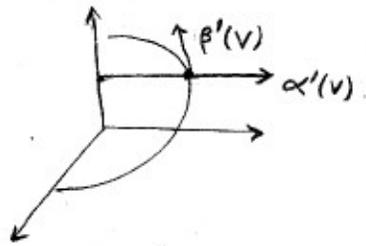
$$\beta'(v) = \langle -u_0 \sin v, u_0 \cos v, 1 \rangle$$

We want a point of intersection

$$\alpha(\tilde{u}) = \beta(\tilde{v})$$

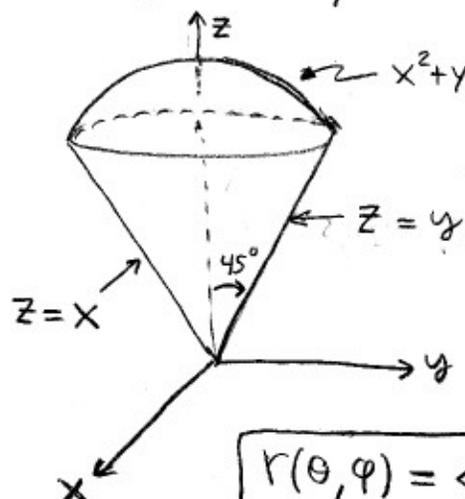
$$\langle \tilde{u} \cos v_0, \tilde{u} \sin v_0, v_0 \rangle = \langle u_0 \cos \tilde{v}, u_0 \sin \tilde{v}, \tilde{v} \rangle$$

obviously we should choose $\tilde{u} = u_0$ and $\tilde{v} = v_0$ so $\alpha(u_0) = \beta(v_0) = r(u_0, v_0)$,
Note then $\alpha'(u_0) \cdot \beta'(v_0) = 0 \therefore$ the coordinate curves are \perp .



• the text's picture I.) is a better rendition.

§10.5#21 Find a parametrization of $x^2 + y^2 + z^2 = 4$ above $z = \sqrt{x^2 + y^2}$



Use spherical coordinates.

$$x = 2 \cos \theta \sin \varphi$$

$$y = 2 \sin \theta \sin \varphi$$

$$z = 2 \cos \varphi$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \pi/4$$

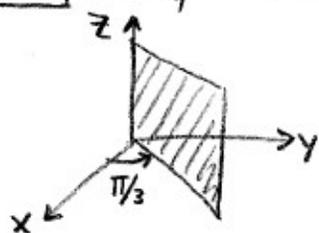
$$\mathbf{r}(\theta, \varphi) = \langle 2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi \rangle \quad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/4 \end{array}$$

• there are many other equally correct answers, parametrizations aren't unique.

§9.7#12 The eqⁿ $\rho = 3$ is all $(x, y, z) \in \mathbb{R}^3$ such that

$\rho = \sqrt{x^2 + y^2 + z^2} = 3 \Rightarrow x^2 + y^2 + z^2 = 9$, this is a sphere of radius 3 centered at the origin.

§9.7#14 The eqⁿ $\theta = \pi/3$ is a half-plane. It has Cartesian eqⁿ's



$$\tan \theta = y/x \quad \therefore \tan(\pi/3) = y/x$$

$$\therefore \boxed{\sqrt{3}x - y = 0} \quad x, y > 0$$

plane with normal $\langle \sqrt{3}, -1, 0 \rangle$.

§9.7#16 $\rho \sin \phi = 2$

$$\rho^2 \sin^2 \phi = \rho^2 (1 - \cos^2 \phi) = 4 = \rho^2 - (\rho \cos \phi)^2$$

$$\Rightarrow 4 = x^2 + y^2 + z^2 - z^2 \Rightarrow \boxed{4 = x^2 + y^2}$$

this is a cylinder with radius 2 and axis the z -axis.

Alternatively,

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$\Rightarrow x = 2 \cos \theta$$

$$\Rightarrow y = 2 \sin \theta$$

$$\Rightarrow x^2 + y^2 = 4(\cos^2 \theta + \sin^2 \theta) = 4$$

• you may have found a completely different argument altogether.

§9.7#19 $r^2 + z^2 = x^2 + y^2 + z^2 = \rho^2 = 25$. This is a sphere centered at the origin with radius 5.

§9.7#21 Write $z = x^2 + y^2$ in cylindricals and sphericals.

$$z = (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) = (\rho \sin \phi)^2$$

and $z = \rho \cos \phi \therefore \cos \phi = \rho \sin^2 \phi$. In cylindricals,

$$x^2 + y^2 = (r \cos \theta)^2 + (r \sin \theta)^2 = r^2 = z$$

Remark: beware the other conventions. I usually use (r, ϕ, θ) in place of (ρ, θ, ϕ) and I like to use $s = \sqrt{x^2 + y^2}$ and ϕ for polar coordinates. These conventions are common in physics for example. One excellent presentation of those conventions and vector calculus as it applies to Electrodynamics is Griffith's Intro to E&M text. I will behave in this course and use the inferior math conventions. (mostly)

§12.9#1 Find the Jacobian of the transformation

$$x = u + 4v \quad \text{and} \quad y = 3u - 2v$$

By definition,

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = -2 - 12 = -14$$

§12.9#4 Let $x = \alpha \sin \beta$ and $y = \alpha \cos \beta$ find the Jacobian of $(x,y) \mapsto (\alpha,\beta)$.

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} \equiv \det \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{bmatrix} = \det \begin{bmatrix} \sin \beta & \alpha \cos \beta \\ \cos \beta & -\alpha \sin \beta \end{bmatrix} = -\alpha \sin^2 \beta - \alpha \cos^2 \beta$$

$$\therefore \frac{\partial(x,y)}{\partial(\alpha,\beta)} = -\alpha$$

§12.9#6 Let $x = e^{u-v}$, $y = e^{u+v}$, $z = e^{u+v+w}$ find Jacobian,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

$$= \det \begin{bmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{bmatrix}$$

$$= e^{u+v+w} (e^{u-v} e^{u+v} + e^{u+v} e^{u-v})$$

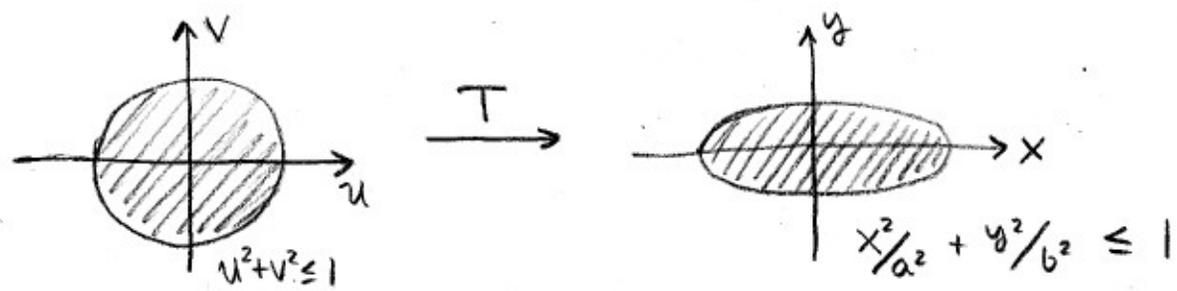
$$= e^{u+v+w} (e^{2u} + e^{2u})$$

$$= \boxed{2e^{3u+v+w}}$$

§12.9#10 $S = \{(u,v) \mid u^2 + v^2 \leq 1\}$ and $x = au$, $y = bv$

notice $x^2/a^2 + y^2/b^2 = a^2 u^2/a^2 + b^2 v^2/b^2 = u^2 + v^2 \leq 1$

thus if we define $T(u,v) = (au, bv)$ we find

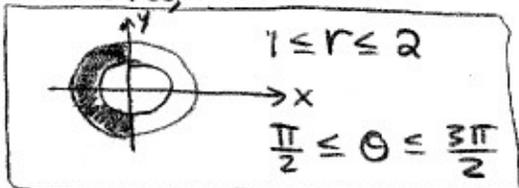


the transformation T deforms a disk to an oval.

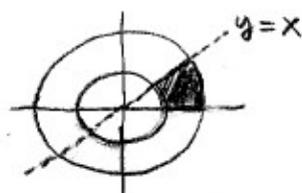
Remark: I have changed the general ordering of the sections in Stewart. My hope was to make the treatment more logically ordered. For example, since we have already completed the Jacobians we know how to change coordinates in a double, triple, etc... integration. In these solⁿ's I will assume a few results from our lecture. I may expect you to prove those results on the test/final. (Don't worry I'll tell you what if any "proofs" are on our tests.)

§12.4#10 Evaluate the following integral in polar coordinates,

$$\begin{aligned} \iint_R (x+y) dA &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 (r\cos\theta + r\sin\theta) r dr d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\cos\theta + \sin\theta) d\theta \int_1^2 r^2 dr \\ &= \left[\sin\theta - \cos\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{1}{3} r^3 \right]_1^2 \\ &= (-1-1) \left(\frac{1}{3}(8-1) \right) = \boxed{-\frac{14}{3}} \end{aligned}$$



§12.4#13 Let $R = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$



$$\begin{aligned} R &= \{(r,\theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{4}\} \\ \tan^{-1}(y/x) &= \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta. \end{aligned}$$

$$\begin{aligned} \iint_R \tan^{-1}(y/x) dA &= \int_0^{\pi/4} \int_1^2 \theta r dr d\theta \\ &= \frac{1}{2} \theta^2 \Big|_0^{\pi/4} \left[\frac{1}{2} r^2 \right]_1^2 \\ &= \frac{1}{4} \left(\frac{\pi^2}{16} \right) (4-1) \\ &= \boxed{\frac{3\pi^2}{64}} \end{aligned}$$

notice that since the integrand can be factored into a purely θ and purely r dependent portion AND the bounds are constants we can just multiply the r -part times the θ -part.

§12.4#16 Find volume bounded by $z = 18 - 2x^2 - 2y^2$ and $z = 0$. The double integral will yield the volume, first convert to polars.

$$z = 18 - 2r^2 \cos^2 \theta - 2r^2 \sin^2 \theta = 18 - 2r^2$$

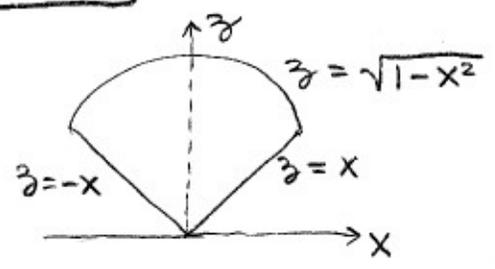
the surface intersects $z = 0$ to give bounds on x, y or better yet r, θ .

$$z = 0 = 18 - 2r^2 \Rightarrow 9 = r^2 \Rightarrow \boxed{r = 3}$$

the surface intersects $z = 0$ in a circle $r = 3$, $R = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$

$$\begin{aligned}
 V &= \iint_R z \, dA = \int_0^{2\pi} \int_0^3 (18 - 2r^2) r \, dr \, d\theta \\
 &= (2\pi) \left(9r^2 - \frac{2}{4} r^4 \right) \Big|_0^3 \\
 &= (2\pi) \left[81 - \frac{1}{2}(81) \right] \\
 &= \boxed{81\pi}
 \end{aligned}$$

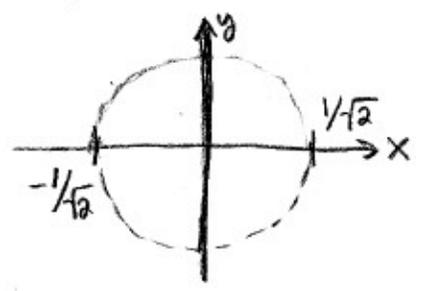
§12.4#19 Find volume bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 1$



points of intersection have (on $y = 0$ slice)

$$\begin{aligned}
 \sqrt{1-x^2} &= x \\
 1-x^2 &= x^2 \\
 1 &= 2x^2 \therefore x = \pm 1/\sqrt{2}
 \end{aligned}$$

the volume is obtained by rotating our picture about the z -axis. I give the top-view of the surface notice $0 \leq r \leq 1/\sqrt{2}$ & $0 \leq \theta \leq 2\pi$.



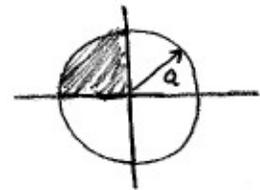
Since our shape goes between the cone and sphere we cannot just integrate z . Instead think a little.

$$\begin{aligned}
 dV &= (z_{\text{top}} - z_{\text{bottom}}) \, dA \quad \leftarrow \text{(typical infinitesimal volume.)} \\
 &= (\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}) \, dx \, dy \\
 &= (\sqrt{1-r^2} - r) \, r \, dr \, d\theta
 \end{aligned}$$

$$\begin{aligned}
 V = \int dV &= \int_0^{2\pi} \int_0^{1/\sqrt{2}} (r\sqrt{1-r^2} - r^2) \, dr \, d\theta = 2\pi \left(-\frac{1}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3 \right) \Big|_0^{1/\sqrt{2}} \\
 &= -\frac{2\pi}{3} \left(\left(\frac{1}{2}\right)^{3/2} + \left(\frac{1}{\sqrt{2}}\right)^3 - 1 \right) = \boxed{\frac{\pi}{3} (2 - \sqrt{2})}
 \end{aligned}$$

§12.4 #26

$$\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2 y \, dx \, dy \rightarrow \begin{matrix} 0 \leq y \leq a \\ -\sqrt{a^2-y^2} \leq x \leq 0 \\ x^2+y^2=a^2 \\ \text{(the left half)} \end{matrix}$$



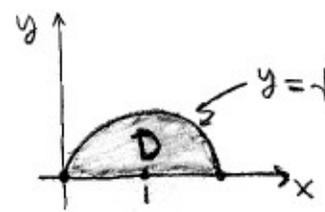
$$\begin{matrix} 0 \leq r \leq a \\ \frac{\pi}{2} \leq \theta \leq \pi \end{matrix}$$

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} \int_0^a (r^2 \cos^2 \theta)(r \sin \theta) r \, dr \, d\theta &= \int_{\frac{\pi}{2}}^{\pi} \cos^2 \theta \sin \theta \, d\theta \int_0^a r^4 \, dr \\ &= \left(-\frac{1}{3} \cos^3 \theta \Big|_{\frac{\pi}{2}}^{\pi} \right) \left(\frac{a^5}{5} \right) \\ &= -\frac{1}{3}(-1) \left(\frac{a^5}{5} \right) = \boxed{\frac{a^5}{15}} \end{aligned}$$

§12.4 #28

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx$$

$$\begin{matrix} 0 \leq y \leq \sqrt{2x-x^2} = \sqrt{x(2-x)} \\ 0 \leq x \leq 2 \end{matrix}$$



$$\begin{aligned} y = \sqrt{2x-x^2} &\Rightarrow y^2 = 2x-x^2 \\ &\Rightarrow x^2-2x+y^2=0 \\ &\Rightarrow (x-1)^2+y^2=1 \end{aligned}$$

(it's a circle of radius one at (1,0))

It should be clear that D has $0 \leq \theta \leq \pi/2$.
 It is also clear that the bound on r must depend on θ since we have differing radii for differing θ (for example $r=2$ $\theta=0$ while $r=0$ for $\theta=\pi/2$). We need to convert $y = \sqrt{2x-x^2}$ to a more useful form for us,

$$r^2 = x^2+y^2 = x^2+2x-x^2 = 2x = 2r \cos \theta \Rightarrow \underline{r=2 \cos \theta}$$

this checks with the limiting cases ($r=0 = \cos(\pi/2)$ & $r=2 = 2 \cos(0)$)

$$\begin{aligned} \iint_D \sqrt{x^2+y^2} \, dA &= \int_0^{\pi/2} \int_0^{2 \cos \theta} r^2 \, dr \, d\theta \\ &= \int_0^{\pi/2} \frac{8}{3} \cos^3 \theta \, d\theta \\ &= \int_0^{\pi/2} \frac{8}{3} (1-\sin^2 \theta) \cos \theta \, d\theta \\ &= \frac{8}{3} \left(\sin \theta - \frac{1}{3} \sin^3 \theta \Big|_0^{\pi/2} \right) = \frac{8}{3} \left(1 - \frac{1}{3} \right) = \boxed{\frac{16}{9}} \end{aligned}$$

§12.4#32 Let D_a be disk of radius a centered at origin, define

$$\begin{aligned}
 \text{a.) } I &= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dA = \lim_{a \rightarrow \infty} \iint_{D_a} e^{-x^2-y^2} dA \\
 &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta \\
 &= \lim_{a \rightarrow \infty} \left[(2\pi) \frac{-1}{2} e^{-r^2} \Big|_0^a \right] \\
 &= \lim_{a \rightarrow \infty} \pi(-e^{-a^2} + 1) \\
 &= \pi
 \end{aligned}$$

b.) Let $S_a = [-a, a] \times [-a, a]$

$$\begin{aligned}
 I &= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dA = \lim_{a \rightarrow \infty} \iint_{S_a} e^{-x^2-y^2} dA \\
 &= \lim_{a \rightarrow \infty} \int_{-a}^a \int_{-a}^a e^{-x^2} e^{-y^2} dx dy \\
 &= \lim_{a \rightarrow \infty} \left(\int_{-a}^a e^{-x^2} dx \right) \left(\int_{-a}^a e^{-y^2} dy \right) \\
 &= \lim_{a \rightarrow \infty} \left(\int_{-a}^a e^{-x^2} dx \right) \lim_{a \rightarrow \infty} \left(\int_{-a}^a e^{-y^2} dy \right) \\
 &= \left[\lim_{a \rightarrow \infty} \left(\int_{-a}^a e^{-x^2} dx \right) \right]^2 = \pi
 \end{aligned}$$

• I'll let you finish it. (there's not much left.)

§12.4#33 Notice $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ follows from #32.

Assume that $\frac{d}{da} \int_{-\infty}^{\infty} e^{-ax^2} dx = \int_{-\infty}^{\infty} \frac{d}{da} (e^{-ax^2}) dx$.

This will give you many integrals of the form

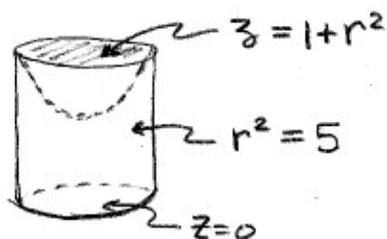
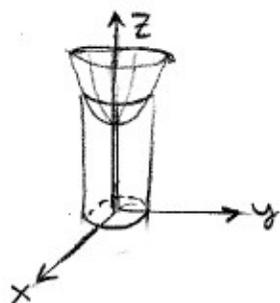
$$\int_{-\infty}^{\infty} z^n e^{-az^2} dz = \text{nice-formula}$$

But we cannot always do this the details are technical

§12.8#6 Clearly $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/2$ and $\pi/2 \leq \theta \leq 2\pi$
 thus an integration over the pictured region would be,

$$\int_1^2 \int_0^{\pi/2} \int_{\pi/2}^{2\pi} f(\rho, \phi, \theta) d\theta d\phi d\rho$$

§12.8#9 $E \subseteq \mathbb{R}^3$ enclosed by $z = 1 + x^2 + y^2$, $x^2 + y^2 = 5$ and $z = 0$



$$\begin{aligned} 0 &\leq z \leq 1+r^2 \\ 0 &\leq r \leq \sqrt{5} \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

$$\begin{aligned} \iiint_E e^z dV &= \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z r dz dr d\theta \\ &= (2\pi) \int_0^{\sqrt{5}} r(e^{1+r^2} - e^0) dr \\ &= 2\pi \left(\frac{1}{2} e^{1+r^2} \Big|_0^{\sqrt{5}} - \frac{1}{2} r^2 \Big|_0^{\sqrt{5}} \right) \\ &= \boxed{\pi(e^6 - e - 5)} \end{aligned}$$

§12.8#18 E is region with $0 \leq \rho \leq 1$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$,

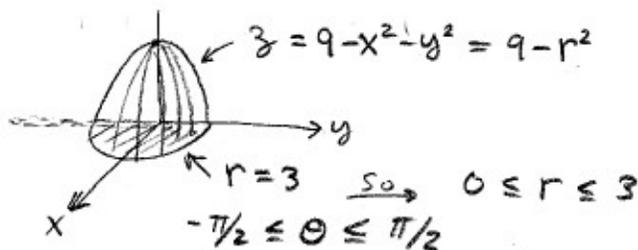
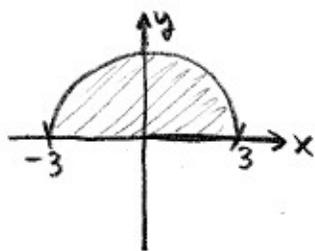
$$\begin{aligned} \iiint_E (x^2 + y^2) dV &= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta) \rho^2 \sin \phi d\theta d\phi d\rho \\ &= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho^4 \sin^3 \phi d\theta d\phi d\rho \\ &= \int_0^1 \rho^4 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} (1 - \cos^2 \phi) \sin \phi d\phi \\ &= \frac{1}{5} \cdot 2\pi \cdot \left(\frac{1}{3} \cos^3 \phi - \cos \phi \Big|_0^{\pi/2} \right) \\ &= \frac{2\pi}{5} \left(\frac{1}{3} \cancel{\cos^3 \frac{\pi}{2}} - \cancel{\cos \frac{\pi}{2}} - \frac{1}{3} \underbrace{\cos^3(0) + \cos(0)}_{2/3} \right) \\ &= \boxed{\frac{4\pi}{15}} \end{aligned}$$

§12.8#32

H83

$$I = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} dz dy dx \Rightarrow \begin{aligned} -3 \leq x \leq 3 \\ 0 \leq y \leq \sqrt{9-x^2} \\ 0 \leq z \leq 9-x^2-y^2 \end{aligned}$$

$$z = 9 - x^2 - y^2 = 9 - r^2, \quad y^2 = 9 - x^2 \Rightarrow r^2 = 9$$



$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{9-r^2} r^2 dz dr d\theta \quad : \text{ using } dV = r dr d\theta dz \\ &\quad \text{and } \sqrt{x^2+y^2} = \sqrt{r^2} = r \\ &= \int_{-\pi/2}^{\pi/2} \int_0^3 (9r^2 - r^4) dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \left(3r^3 - \frac{1}{5}r^5 \right) \Big|_0^3 d\theta \\ &= \pi \left(81 - \frac{1}{5}(243) \right) = \left(\frac{405 - 243}{5} \right) \pi = \boxed{\frac{162\pi}{5}} \end{aligned}$$

§12.8#36 Suppressing the limit notation,

$$\begin{aligned} \iiint_{\mathbb{R}^3} \rho e^{-\rho^2} dV &= \int_0^{2\pi} \int_0^\pi \int_0^\infty \rho^3 e^{-\rho^2} \sin\phi d\rho d\phi d\theta \\ &= (2\pi)(-\cos\pi + \cos(0)) \int_0^\infty \rho^3 e^{-\rho^2} d\rho \end{aligned}$$

Use integration by parts twice.

$$\begin{aligned} \int \underbrace{z^2}_u \underbrace{ze^{-z^2}}_{dV} dz &= z^2 \left(-\frac{1}{2} e^{-z^2} \right) + \int \frac{1}{2} e^{-z^2} 2z dz = -\frac{z^2}{2} e^{-z^2} + \int ze^{-z^2} dz \\ &= -\frac{z^2}{2} e^{-z^2} - \frac{1}{2} e^{-z^2} + C \end{aligned}$$

$$\therefore \int_0^\infty \rho^3 e^{-\rho^2} d\rho = \frac{1}{2} \left(z^2 e^{-z^2} + e^{-z^2} \right) \Big|_0^\infty = \frac{1}{2} \quad \left(\text{using l'Hopital's on } z^2 e^{-z^2} \right)$$

$$\therefore \iiint_{\mathbb{R}^3} \rho e^{-\rho^2} dV = 4\pi/a = \boxed{2\pi}$$