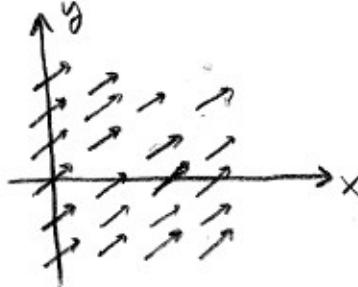
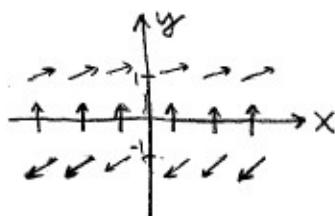


§13.1#1 Sketch the vector field $F(x, y) = \frac{1}{2}(\hat{i} + \hat{j}) = \langle \frac{1}{2}, \frac{1}{2} \rangle$



this one is pretty boring.
those are supposed to be
equally space arrows pointing
in the $\langle \frac{1}{2}, \frac{1}{2} \rangle$ direction.

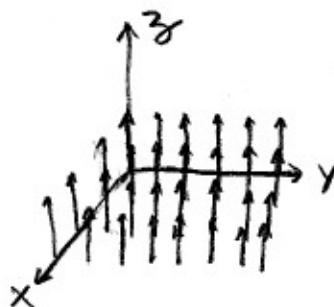
§13.1#3 Sketch the vector field $F = \langle y, \frac{1}{2} \rangle$.



y	F
0	$\langle 0, \frac{1}{2} \rangle = \frac{1}{2}\langle 0, 1 \rangle$
1	$\langle 1, \frac{1}{2} \rangle = \frac{1}{2}\langle 2, 1 \rangle$
-1	$\langle -1, \frac{1}{2} \rangle = \frac{1}{2}\langle -2, 1 \rangle$

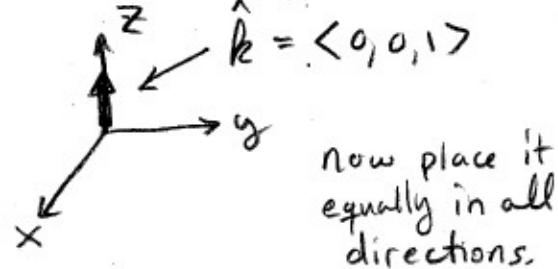
Remark: these are much better left to Maple, I mean all we're doing is giving a graphical representation of a table of values. It's neat and all, but inconvenient by hand.

§13.1#7 $F = \langle 0, 0, 1 \rangle$



(Customarily for constant vector fields one vector is a sufficient description)

well, you
get the
idea,



§13.1#15 $F = \langle 1, 2, 3 \rangle$ match with plot on p. 911, it's
clearly IV since that is the only constant vector field.

Remark: the reason for doing these early exercises is not for you to get good at graphing vector fields. Rather, the point is for you to think about visualizing them. Usually you'll need to imagine the field in your mind's eye then do some analysis in view of the field's influence.

§13.1 #21 Let $f(x, y) = \ln(x+2y)$ then $\nabla f = \left\langle \frac{1}{x+2y}, \frac{2}{x+2y} \right\rangle$. H85

§13.1 #24 Let $f(x, y, z) = x \cos(y/z)$

$$\begin{aligned}\nabla f &= \left\langle \cos(y/z), -x \sin(y/z) \frac{\partial}{\partial y} \left(\frac{y}{z} \right), -x \sin(y/z) \frac{\partial}{\partial z} \left(\frac{y}{z} \right) \right\rangle \\ &= \left\langle \cos(y/z), -(x/z) \sin(y/z), xy/z^2 \sin(y/z) \right\rangle\end{aligned}$$

§13.1 #27 $f(x, y) = \sin(x) + \sin(y)$

$$\nabla f = \langle \cos(x), \cos(y) \rangle$$

Remark: my methods here are somewhat obtuse, maple would provide nicer solⁿ. Unfortunately for me I've no Maple on vacation.

Consider that,

$$y = x + (2n-1)\pi \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow \sin(x) + \sin(y) = \sin(x) + \sin(x + (2n-1)\pi)$$

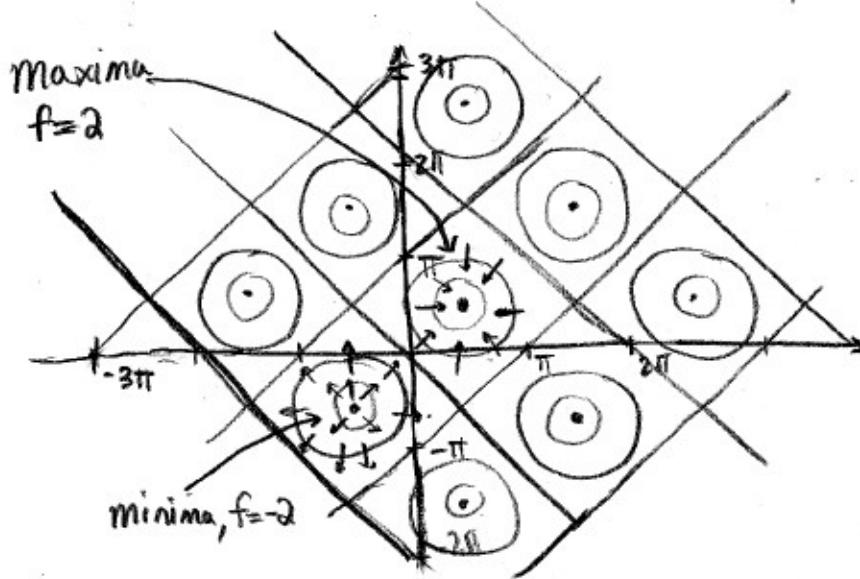
$$= \sin(x) + \sin(x) \cos(2n\pi - \pi) + \cos(x) \sin((2n-1)\pi)$$

$$= \sin(x) + \sin(x) \cos(-\pi)$$

$$= \sin(x) - \sin(x)$$

$$= 0 \quad \Rightarrow \quad y = x + (2n-1)\pi \text{ give } f=0 \text{ contours}$$

$$y = x \pm \pi, y = x \pm 3\pi, \dots$$



Notice $\nabla f = \langle \cos(x), \cos(y) \rangle = 0$ when $f=\pm 2$.

Lets find a few contours,
 $\sin(x) + \sin(y) = 0$

$$\sin(x) = -\sin(y)$$

$$x=0 \Rightarrow 0 = -\sin(y) \Rightarrow y = n\pi$$

$$x=\pi/2 \Rightarrow 1 = -\sin(y) \Rightarrow y = -\frac{\pi}{2} + \frac{3\pi}{2} \text{ some } n.$$

Lets see in other words,

$$y = x + \pi$$

$$\sin(y) = \sin(x + \pi)$$

$$= \sin(x) \cos(\pi) + \sin(\pi) \cos(x)$$

$$= -\sin(x).$$

Notice that

$$y = -x + 2n\pi$$

also yields $f=0$

the "•" are where
 $f(x, y) = 2 \text{ or } -2$

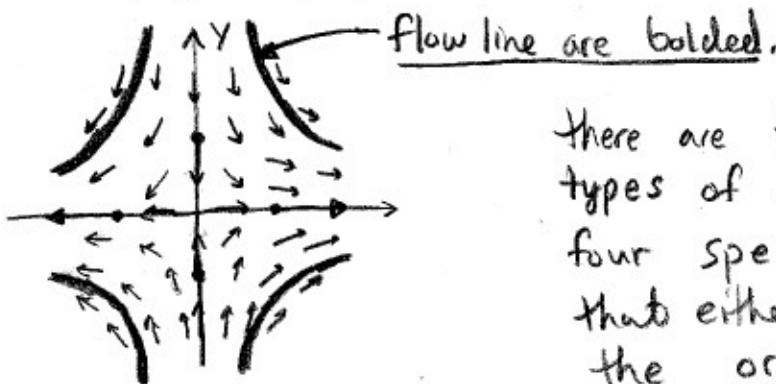
this graphed in 3-d
gives a bumpy plane.

The gradient is
 \perp to the level
surfaces.

§13.1 #35 The flow lines (or stream lines or integral curves more generally) of a vector field are the paths followed by a particle whose velocity field is the given vector field. In other words given $F(x, y)$ a vector field then $\vec{r}(t) = \langle x(t), y(t) \rangle$ the parametrization of a flow line if

$$\vec{r}'(t) = F(\vec{r}(t)) \text{ aka } \langle x'(t), y'(t) \rangle = F(x(t), y(t))$$

a.) Sketch $F(x, y) = \langle x, -y \rangle$



there are four disconnected types of flow lines, and four special flow lines that either go to leave the origin.

b.) Find the flow lines. That is solve

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = F(x, y) = \langle x, -y \rangle$$

This gives two differential eq's,

$$\frac{dx}{dt} = x \quad \text{and} \quad \frac{dy}{dt} = -y$$

$$\lambda = 1$$

$$x = C_1 e^t$$

$$\lambda = -1$$

$$y = C_2 e^{-t}$$

thus the flow lines are $\vec{r}(t) = \langle C_1 e^t, C_2 e^{-t} \rangle$.

- I'll let you find the particular values of C_1 and C_2 that encourage the flow line to pass through $(1, 1)$.

Remark: for those of you who have parameter phobia I mention that these curves are also expressed as, supposing $C_1, C_2 \neq 0$.

$$\frac{1}{e^t} = e^{-t} = \frac{x}{C_1} = \frac{C_2}{y} \quad \therefore \boxed{y = \frac{C_1 C_2}{x}}$$

of course $C_1 = 0, C_2 = 0$ yield the special flow lines and other cases follow from whether $C_1 < 0, C_1 > 0, C_2 < 0$ or $C_2 > 0$ etc...

§13.5#1 Let $\mathbf{F} = \langle xy\hat{z}, 0, -x^2y \rangle$

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (xy\hat{z}\hat{i} - x^2y\hat{k}) \\ &= \hat{i} \times \hat{k} \frac{\partial}{\partial x}(-x^2y) + \hat{j} \times \hat{i} \frac{\partial}{\partial y}(xy\hat{z}) + \hat{j} \times \hat{k} \frac{\partial}{\partial y}(-x^2y) + \hat{k} \times \hat{i} \frac{\partial}{\partial z}(xy\hat{z}) \\ &= (-\hat{j})(-2xy) - \hat{k}(x^3) + \hat{i}(-x^2) + \hat{j}(xy) \\ &= \boxed{\langle -x^2, 3xy, -x^3 \rangle} = \nabla \times \mathbf{F}\end{aligned}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy\hat{z}) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-x^2y) = \boxed{yz} = \nabla \cdot \mathbf{F}$$

Remark: technically $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$ is an operator since it operates on functions. Ok, actually it is a vector of operators, this means we can take the cross or dot product of ∇ or just multiply by a function. These three distinct uses give the gradient, curl and divergence. For now we content ourselves to work in Cartesian coordinates where $\frac{\partial}{\partial x} \hat{i}$ and $\hat{i} \frac{\partial}{\partial x}$ need not be distinguished. If we work in other coordinates then usually the unit-coordinate vectors analogous to $\hat{i}, \hat{j}, \hat{k}$ gain a coordinate dependence so we need to be more careful. We'll begin with the easy case.

§13.5#4 Let $\mathbf{F} = \langle 0, \cos(xy), -\sin(xy) \rangle$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \cos(xy) & -\sin(xy) \end{vmatrix} = \boxed{\langle -x\cos(xy) + x\sin(xy), y\cos(xy), -z\sin(xy) \rangle}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial y}(\cos(xy)) + \frac{\partial}{\partial z}(-\sin(xy)) = \boxed{0} = \nabla \cdot \mathbf{F}$$

§13.5 #12] Determine if $\mathbf{F} = \langle 3z^2, \cos(y), 2xz \rangle$ is conservative. If \mathbf{F} was conservative then $\exists f$ such that $\mathbf{F} = \nabla f$. Then in that case $\nabla \times \mathbf{F} = \nabla \times \nabla f = 0$. Hence if $\nabla \times \mathbf{F} \neq 0$ then \mathbf{F} is not conservative. Use notation $\mathbf{F} = \langle F_x, F_y, F_z \rangle$

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle \partial_y F_z - \partial_z F_y, \partial_z F_x - \partial_x F_z, \partial_x F_y - \partial_y F_x \rangle \\ &= \langle 0, 6z - 2z, 0 \rangle \\ &= \langle 0, 4z, 0 \rangle \neq 0 \quad \therefore \boxed{\mathbf{F} \text{ not conservative}}\end{aligned}$$

Remark: there are many times when F_x does not mean $F_x = \frac{\partial F}{\partial x}$, rather it is just the x -component of \mathbf{F} ; $F_x = \mathbf{F} \cdot \hat{i}$. Hopefully it will be clear from the context, I use both notations in this course.

§13.5 #13] Let $\mathbf{F} = \langle 2xy, x^2 + 2yz, y^2 \rangle$. Consider that

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle \partial_y F_z - \partial_z F_y, \partial_z F_x - \partial_x F_z, \partial_x F_y - \partial_y F_x \rangle \\ &= \langle 2y - 2y, 0 - 0, 2x - 2x \rangle = 0.\end{aligned}$$

Further note domain (\mathbf{F}) = \mathbb{R}^3 and the component functions have continuous partial derivatives \therefore by Thm 13.4 on p. 942 \mathbf{F} is conservative. Let's find the scalar function f such that $\mathbf{F} = \nabla f$. We need f to satisfy

$$\begin{aligned}\partial_x f &= 2xy \\ \partial_y f &= x^2 + 2yz \\ \partial_z f &= y^2\end{aligned}$$

Clearly the function $f = x^2y + y^2z + C$ solves the eq's above and hence $\mathbf{F} = \nabla f$.

Remark: most of the time I can see the solⁿ by inspection. Suppose that I/you don't see it (or I ask you to show how to find it) then there is a procedure. We simply integrate several times, the interesting feature is that our "constants" are

$$\begin{aligned}\partial_x f &= 2xy \Rightarrow f = \int 2xy \, dx = x^2y + C_1(y, z) \\ \partial_y f &= x^2 + 2yz = \partial_y(x^2y + C_1) = x^2 + \partial_y C_1 \\ \Rightarrow 2yz &= \partial_y C_1 \Rightarrow C_1 = \int 2yz \, dy = 3y^2 + C_2(z)\end{aligned}$$

functions w.r.t. the un-integrated variables)

then plug this into $\partial_z f = y^2$

§13.5 #13 continued We know $f = x^2y + C_1 = x^2y + 3y^2 + C_2$ and furthermore we know that C_2 is only a fnct. of z .

$$\partial_z f = y^2 = \partial_z(x^2y + 3y^2 + C_2) = y^2 + \partial_z C_2 \therefore \partial_z C_2 = 0$$

But C_2 is only a fnct. of z thus $\frac{\partial C_2}{\partial z} = \frac{dC_2}{dz} = 0 \therefore C_2$ constant.

Therefore, $f = x^2y + y^2z + C_2$

§13.5 #16 $F = \langle y \cos(xy), x \cos(xy), -\sin(z) \rangle$. We could check $\nabla \times F = 0$ and observe the partials of the component fncts. are continuous $\therefore f$ exists so that $\nabla f = F$. However, another approach is just to look for f directly, if it d.n.e then the procedure we used in #13 will reach a contradiction.

Suppose $\exists f$ such that $F = \nabla f$, we need

$$\partial_x f = y \cos(xy)$$

$$\partial_y f = x \cos(xy)$$

$$\partial_z f = -\sin(z)$$

Let us begin,

$$f = \int y \cos(xy) \partial x = \sin(xy) + C_1(y, z)$$

$$\partial_y f = x \cos(xy) = \partial_y (\sin(xy) + C_1) = x \cos(xy) + \partial_y C_1$$

$$\therefore \partial_y C_1 = 0 \Rightarrow C_1 \text{ only fnct. of } z.$$

Finally apply $\partial_z f = -\sin(z)$,

$$\partial_z f = -\sin(z) = \partial_z (\sin(xy) + C_1) = \partial_z C_1$$

$$\Rightarrow \frac{dC_1}{dz} = -\sin(z) \Rightarrow C_1 = \cos(z) + C_2$$

$\therefore f = \sin(xy) + \cos(z) + C_2$

(Sorry.)

we don't count constant fncts as fncts in this terminology.
But, technically $f(x) = 3$ is a fnct of x

Remark: we found f so that $F = \nabla f$. Hence F conservative. The $\nabla \times F = 0$ test is nice, but not required.

§13.5 #18 Can we find G on \mathbb{R}^3 such that

$$\nabla \times G = \langle yz, xy, xy \rangle$$

well it is known that $\nabla \cdot (\nabla \times G) = 0$ for any G . Consider

$$\nabla \cdot (\nabla \times G) = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(xy) = xy \neq 0$$

therefore there is no such vector field G .

§13.5 #20 Show that any vector field of the following form is incompressible

$$F = f(y, z)\hat{i} + g(x, z)\hat{j} + h(x, y)\hat{k}$$

we need $\nabla \cdot F = 0$. This follows from the coordinate functions independence of x for \hat{i} , y for \hat{j} and z for \hat{k}

$$\nabla \cdot F = \frac{\partial}{\partial x}(f(y, z)) + \frac{\partial}{\partial y}(g(x, z)) + \frac{\partial}{\partial z}(h(x, y)) = 0 + 0 + 0 = 0.$$

Remark: In the exercises that follow I find it convenient to use the Einstein index notation. You are not req'd to understand the repeated index notation, instead you may employ the longer brute-force method.

Curl, Div & Grad in the repeated index notation

$$\nabla f = e_i \partial_i f$$

$$\nabla \times F = e_i \epsilon_{ijk} \partial_j F_k$$

$$\nabla \cdot F = \partial_i F_i$$

where $F = F_i e_i$ is a vector field and f is a scalar funct.

§13.5 #21 Let $F = F_i e_i$ and $G = G_i e_i$ as usual,

$$\begin{aligned}\nabla \cdot (F+G) &= \partial_i [(F+G)_i] \\ &= \partial_i [F_i + G_i] \xrightarrow{(F+G)_i = F_i + G_i} \text{this is how we add vectors.} \\ &= \partial_i F_i + \partial_i G_i : \text{using linearity of partial differentiation w.r.t. } x^i. \\ &= \nabla \cdot F + \nabla \cdot G //.\end{aligned}$$

$$\therefore \boxed{\nabla \cdot (F+G) = \nabla \cdot F + \nabla \cdot G}$$

§13.5 #22

$$\begin{aligned}
 \nabla \times (F+G) &= e_i \epsilon_{ijk} \partial_j [(F+G)_k] \\
 &= e_i \epsilon_{ijk} \partial_j [F_k + G_k] : \text{def}^{\text{a}} \text{ of vector addition, its componentwise.} \\
 &= e_i \epsilon_{ijk} \partial_j F_k + e_i \epsilon_{ijk} \partial_j G_k : \text{linearity of partial differentiation.} \\
 &= \nabla \times F + \nabla \times G \quad \therefore \boxed{\nabla \times (F+G) = \nabla \times F + \nabla \times G} //
 \end{aligned}$$

§13.5 #23 Let $F = F_i e_i$ and f a scalar function.

$$\begin{aligned}
 \nabla \cdot (fF) &= \partial_i [(fF)_i] \\
 &= \partial_i [f F_i] : \text{def}^{\text{a}} \text{ of scalar multiplication, its componentwise.} \\
 &= (\partial_i f) F_i + f (\partial_i F_i) : \text{applying product-rule for each } i=1,2,3 \text{ in the implicit summation.} \\
 &= (\nabla f)_i F_i + f (\nabla \cdot F) : \text{notice } \nabla f = e_i \partial_i f \Rightarrow (\nabla f)_i = \partial_i f. \\
 &= \boxed{(\nabla f) \cdot F + f (\nabla \cdot F) = \nabla \cdot (fF)} //
 \end{aligned}$$

§13.5 #24 Again F and f as in #23,

$$\begin{aligned}
 \nabla \times (fF) &= e_i \epsilon_{ijk} \partial_j [(fF)_k] \\
 &= e_i \epsilon_{ijk} \partial_j [f F_k] \\
 &= e_i \epsilon_{ijk} [(\partial_j f) F_k + f \partial_j F_k] \\
 &= e_i \epsilon_{ijk} (\nabla f)_j F_k + e_i \epsilon_{ijk} (\partial_j F_k) f \\
 &= (\nabla f) \times F + (\nabla \times F) f \\
 &= \boxed{(\nabla f) \times F + f (\nabla \times F) = \nabla \times (fF)}
 \end{aligned}$$

Remark: Again I used, the definition of scalar multiplication, and product rule of partial differentiation.

§13.5 #25 Let $F = F_i e_i$ and $G = G_j e_j$.

$$\begin{aligned}
 \nabla \cdot (F \times G) &= \partial_i [(F \times G)_i] \\
 &= \partial_i [\epsilon_{ijk} F_j G_k] \quad \Rightarrow \quad F \times G = e_i \epsilon_{ijk} F_j G_k \\
 &= \epsilon_{ijk} \partial_i (F_j G_k) : \epsilon_{ijk} \text{ is a constant for fixed } i, j, k. \\
 &= \epsilon_{ijk} [\partial_i F_j] G_k + F_j \partial_i G_k : \text{linearity of } \partial_i \\
 &= (\epsilon_{ikj} \partial_i F_j) G_k - F_j (\epsilon_{ijk} \partial_i G_k) : \epsilon_{ijk} = -\epsilon_{jik} \\
 &= (\nabla \times F)_k G_k - F_j (\nabla \times G)_j \\
 &= \boxed{(\nabla \times F) \cdot G - F \cdot (\nabla \times G) = \nabla \cdot (F \times G)}.
 \end{aligned}$$

I explain more in Remark below.

§13.5 #26 Let f and g be scalar functions.

$$\begin{aligned}
 \nabla \cdot (\nabla f \times \nabla g) &= \partial_i [(\nabla f \times \nabla g)_i] \\
 &= \partial_i [\epsilon_{ijk} (\nabla f)_j (\nabla g)_k] \\
 &= \epsilon_{ijk} \partial_i [(\partial_i f)(\partial_k g)] \\
 &= \epsilon_{ijk} [(\partial_i \partial_j f) \partial_k g + (\partial_j f) \partial_i \partial_k g] \\
 &= (\epsilon_{ijk} \partial_i \partial_j f) \partial_k g + (\partial_j f) (\epsilon_{ijk} \partial_i \partial_k g) \\
 &\quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
 &\quad \text{antisym.} \quad \text{symmetric} \quad \text{antisym.} \quad \text{symmetric} \\
 &\quad \text{in } i,j \quad \text{in } i,j \quad \text{in } i,k \quad \text{in } i,k \\
 &= 0 \quad \text{using (iii) of the über Lemma}
 \end{aligned}$$

Remark: the notation $(F)_i$ or $(\nabla f)_i$ or $(F \times G)_i$ means the i^{th} components of those vectors. We can write a formula via the dot-product. We assume $e_i \cdot e_j = \delta_{ij}$ (orthonormal basis)

$$(F)_i = F \cdot e_i = F_m e_m \cdot e_i = F_m \delta_{mi} = F_i$$

$$(\nabla f)_i = \nabla f \cdot e_i = e_i \cdot e_m \partial_m f = \delta_{im} \partial_m f = \partial_i f$$

$$(F \times G)_i = e_i \cdot e_m \epsilon_{mjk} F_j G_k = \delta_{im} \epsilon_{mjk} F_j G_k = \epsilon_{ijk} F_j G_k$$

(the main thing here is that e_i = vector while no e_i 's \Rightarrow scalar.)

§13.5#37 Let $F = F_k e_k$ as usual,

$$\begin{aligned}
 \nabla \times (\nabla \times F) &= e_m \epsilon_{mjk} \partial_j [(\nabla \times F)_k] \\
 &= e_m \epsilon_{mjk} \partial_j [\epsilon_{klj} \partial_l F_i] \\
 &= e_m \epsilon_{mjk} \epsilon_{klj} \partial_j \partial_l F_i \\
 &= e_m (\delta_{ml} \delta_{ji} - \delta_{mi} \delta_{jl}) \partial_j \partial_l F_i \quad] \text{ using the uber lemma part (i).} \\
 &= e_m \delta_{ml} \delta_{ji} \partial_j \partial_l F_i - e_m \delta_{mi} \delta_{jl} \partial_j \partial_l F_i \\
 &= e_m \partial_i \partial_m F_i - e_m \partial_j \partial_j F_m \\
 &= e_m \partial_m \partial_i F_i - \partial_j \partial_j e_m F_m \\
 &= \boxed{\nabla(\nabla \cdot F) - \nabla^2 F = \nabla \times (\nabla \times F)} //.
 \end{aligned}$$

Remark: $\nabla^2 = \partial_j \partial_j = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.

the expression $\nabla^2 F = \langle \nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3 \rangle$, this is what is meant. The identity above is an important one to Electromagnetism, it helps show that E and B satisfy a wave eq^{ns} in empty space, that is E & B are transverse oscillating waves. With a bit of further physical interpretation it shows Maxwell's Eq^{ns} \Rightarrow the speed of light is constant!

§13.5#36 Classical Electromagnetism is $F = q(E + v \times B)$ and the following 4 vector-partial differential eq^{ns} (Maxwell's Eq^{ns})

$$\boxed{\nabla \cdot E = 0 \quad \nabla \cdot B = 0 \quad \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \quad \nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t}}$$

well in free-space they have the form above ($J=0$ and $\rho=0$ no charge)

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = -\nabla^2 E = \nabla \times \left(-\frac{1}{c} \frac{\partial B}{\partial t}\right)$$

$$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B = -\nabla^2 B = \nabla \times \left(\frac{1}{c} \frac{\partial E}{\partial t}\right)$$

But $\frac{\partial}{\partial t}$ commutes with ∇ thus,

$$\nabla \times \left(\frac{1}{c} \frac{\partial E}{\partial t}\right) = \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times E) = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{1}{c} \frac{\partial B}{\partial t}\right) = -\frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} \quad \text{wave eq^{ns}}$$

Likewise, $\nabla \times \left(-\frac{1}{c} \frac{\partial B}{\partial t}\right) = -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \therefore \boxed{\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \quad \& \quad \nabla^2 B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}}$

ask
me
more
if you
wish.