

TEST I : CALCULUS III, ma 242-011, Spring 2007

Clearly box or circle your answers. No graphing calculators, do your own work.

**PROBLEM ONE** Let  $A = \langle 0, 1, 2 \rangle$  and  $B = \langle 3, 4, 5 \rangle$

find the following, put your answers on the blanks provided

$$(a.) \hat{A} = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$$

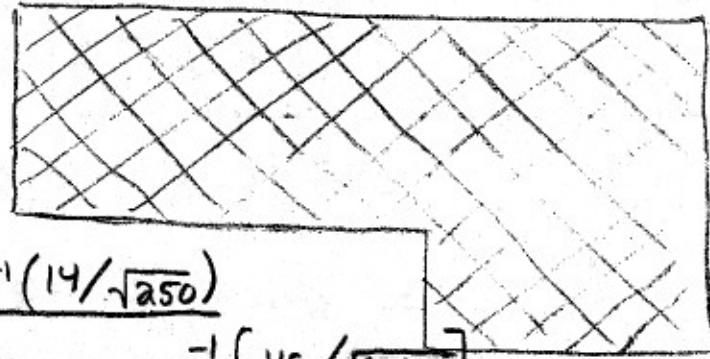
$$\hat{B} = \frac{1}{\sqrt{50}} \langle 3, 4, 5 \rangle$$

$$(b.) A \cdot B = \underline{\quad 14 \quad}$$

$$(c.) \text{angle between } A \text{ & } B = \underline{\cos^{-1}(14/\sqrt{250})}$$

$$(d.) \text{angle between } A+B \text{ and } A-B = \underline{\cos^{-1}[-45/\sqrt{(23)(67)}]}$$

$$(e.) \text{a vector } V \text{ which is } \perp \text{ to } A \text{ & } B \text{ is } V = \underline{\langle -3, 6, -3 \rangle}$$



Show your work below

$$a.) \hat{A} = \frac{A}{|A|} = \frac{\langle 0, 1, 2 \rangle}{\sqrt{5}}$$

$$\hat{B} = \frac{B}{|B|} = \frac{\langle 3, 4, 5 \rangle}{\sqrt{9+16+25}}$$

$$b.) A \cdot B = \langle 0, 1, 2 \rangle \cdot \langle 3, 4, 5 \rangle = 0 + 4 + 10 = 14$$

$$c.) A \cdot B = |A||B| \cos \theta \therefore \theta = \cos^{-1}\left(\frac{A \cdot B}{|A||B|}\right) = \cos^{-1}\left(\frac{14}{\sqrt{5}\sqrt{50}}\right)$$

$$d.) (A+B) \cdot (A-B) = A \cdot A + B \cdot A - A \cdot B - B \cdot B = |A|^2 - |B|^2 = 5 - 50$$

$$\text{Anyway, } A+B = \langle 0, 1, 2 \rangle + \langle 3, 4, 5 \rangle = \langle 3, 5, 7 \rangle$$

$$A-B = \langle 0, 1, 2 \rangle - \langle 3, 4, 5 \rangle = \langle -3, -3, -3 \rangle$$

$$|A+B| = \sqrt{9+25+49} = \sqrt{83}$$

$$|A-B| = \sqrt{9+9+9} = \sqrt{27}$$

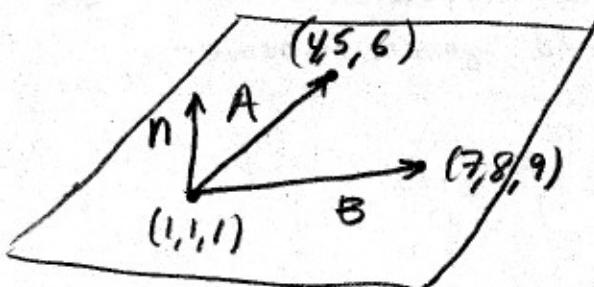
$$\text{Say } (A+B) \cdot (A-B) = |A+B||A-B| \cos \beta \text{ where } \beta \text{ is the angle we desire, then } \beta = \cos^{-1}\left[\frac{-45}{\sqrt{83}\sqrt{27}}\right]$$

(e.) We know  $A \times B$  is  $\perp$  to  $A \# B$ .  
This means  $A \times B$  is the vector we seek,

$$\begin{aligned} A \times B &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} \\ &= \hat{i} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} - \hat{j} \begin{vmatrix} 0 & 2 \\ 3 & 5 \end{vmatrix} + \hat{k} \begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} \\ &= i(5-8) - j(0-6) + k(0-3) \\ &= \langle -3, 6, -3 \rangle. \end{aligned}$$

As a check on our calculation notice that  
 $A \# B$  are  $\perp$  to  $\langle -3, 6, -3 \rangle$ .

**PROBLEM TWO** The points  $(1, 2, 3)$ ,  $(4, 5, 6)$ ,  $(7, 8, 9)$  are on a plane  $P$ . Find the eq<sup>n</sup> for  $P$  in terms of  $x, y, z$ .



$$\text{choose } \mathbf{r}_0 = (1, 1, 1)$$

$$\mathbf{A} = (4, 5, 6) - (1, 1, 1) = \langle 3, 4, 5 \rangle$$

$$\mathbf{B} = (7, 8, 9) - (1, 1, 1) = \langle 6, 7, 8 \rangle$$

$$\mathbf{n} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = \langle 32 - 35, 30 - 24, 21 - 24 \rangle$$

$$\mathbf{n} = \langle -3, 6, -3 \rangle$$

We have a point and the normal, thus the eq<sup>n</sup> of the plane follows,

$$-3(x-1) + 6(y-1) - 3(z-1) = 0$$

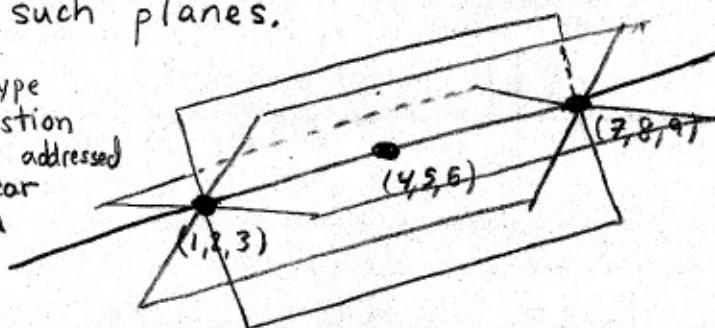
$$-3x + 6y - 3z = -3 + 6 - 3 = 0$$

$$x - 2y + z = 0$$

I'd accept any of these, so long as the eq<sup>n</sup> is the same modulo some algebra.

Remark: It's true the points  $(1, 2, 3)$ ,  $(4, 5, 6)$ ,  $(7, 8, 9)$  lie on a plane, the trouble is there are infinitely many such planes.

- this type of question is best addressed in linear algebra



$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$\mathbf{v} = \langle 3, 3, 3 \rangle$$

any  $\mathbf{n} = \langle a, b, c \rangle \neq 0$  with  $\mathbf{n} \cdot \mathbf{v} = 0$  will do.

$$3a + 3b + 3c = 0$$

has many sol<sup>n</sup>'s.

**PROBLEM THREE** Does  $(x-3)^2 + (y-2)^2 + (z-1)^2 = 4$  intersect  $z=1$ ? If so describe the intersection carefully. Does  $(\star)$  intersect  $x=5$ ? If so describe the intersection carefully. In both cases indicate the algebra behind you're answer.

$$\underline{z=1} \quad (x-3)^2 + (y-2)^2 + (1-1)^2 = 4$$

$$\therefore (x-3)^2 + (y-2)^2 = 4$$

this is a circle of radius 2 centered at  $(3, 2, 1)$  in the  $z=1$  plane.

$$\underline{x=5} \quad (5-3)^2 + (y-2)^2 + (z-1)^2 = 4$$

$$\therefore 4 + (y-2)^2 + (z-1)^2 = 4$$

$$\therefore (y-2)^2 + (z-1)^2 = 0$$

$$\Rightarrow_0 \Rightarrow_0$$

the only way for us to solve this eq<sup>n</sup> is  $y=2$  AND  $z=1$ . Any other real # combination will give something non zero = 0 which is impossible.

The intersection is a point  $(5, 2, 1)$

PROBLEM FOUR Let  $r(t) = \langle 4at^3, 3bt^2, cct \rangle$  then  
calculate  $\int_0^1 r(t) dt$ , show your work.

$$\begin{aligned}\int_0^1 \langle 4at^3, 3bt^2, cct \rangle dt &= \left\langle \int_0^1 4at^3 dt, \int_0^1 3bt^2 dt, \int_0^1 cct dt \right\rangle \\&= \left\langle \frac{4at^4}{4} \Big|_0^1, \frac{3bt^3}{3} \Big|_0^1, \frac{cct^2}{2} \Big|_0^1 \right\rangle \\&= \langle a-0, b-0, c-0 \rangle \\&= \boxed{\langle a, b, c \rangle}\end{aligned}$$

**PROBLEM FIVE** Suppose that  $A$  and  $B$  are arbitrary non-zero vectors. Find vectors  $V$  and  $W$  such that  $A = V + W$  where  $V \parallel B$  and  $W \perp B$ . You must prove that your choices for  $V$  &  $W$  are in fact parallel and perpendicular. Your answer should involve only dot products and unit vectors, components are beside the point.

- We know that  $\text{proj}_B(A)$  is  $\parallel$  to  $B$ . It is the vector projection of  $A$  onto  $B$ . Notice

$$A = \underbrace{A - \text{proj}_B(A)}_W + \underbrace{\text{proj}_B(A)}_V : \text{added zero.}$$

By construction  $V \parallel B$  and we'll see  $W \perp B$ . Let's prove my construction works. Obviously  $A = V + W$ , need to check the  $\parallel \neq \perp$  req'd.

$$V \times B = \text{proj}_B(A) \times B = (A \cdot \hat{B}) \hat{B} \times |B| \hat{B} = (A \cdot B) \hat{B} \times \hat{B}$$

Thus  $V \times B = 0$  showing  $V \parallel B$ .

$$\begin{aligned} W \cdot B &= (A - \text{proj}_B(A)) \cdot B \\ &= A \cdot B - (A \cdot \hat{B}) \hat{B} \cdot B \\ &= A \cdot B - (A \cdot \hat{B}) \hat{B} \cdot (|B| \hat{B}) \\ &= A \cdot B - (A \cdot |B| \hat{B}) \hat{B} \cdot \hat{B}^1 \\ &= A \cdot B - A \cdot B \\ &= 0. \end{aligned}$$

since I told you that  
 $\text{proj}_B(A) \parallel B$  on the formula  
sheet I'll let you slide  
by w/o showing  $V \times B = 0$ .  
You did need to show  
 $W \cdot B = 0$  to prove  
that  $W \perp B$

Thus  $W \perp B$ . So we've shown  $A = V + W$  with  $V \parallel B$  and  $W \perp B$  (provided  $A, B \neq 0$  so that)  
 $\hat{A}$  and  $\hat{B}$  make sense.)

**PROBLEM SIX** Let  $\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$  for  $0 \leq t \leq 2\pi$ .

Find the following quantities, place your answer on the blank provided and place your work after the blanks.

(a.)  $\frac{ds}{dt} = \frac{\sqrt{2}}{\sqrt{2}}$  = speed

(b.)  $s = \frac{\sqrt{2}t}{\sqrt{2}}$  = arc length function

(c.)  $T(t) = \frac{1}{\sqrt{2}} \langle -\sin t, 1, \cos t \rangle$  = unit tangent

(d.)  $N(t) = \frac{\langle -\cos t, 0, -\sin t \rangle}{\sqrt{2}}$  = unit normal

(e.)  $B(t) = \frac{1}{\sqrt{2}} \langle -\sin t, -1, \cos t \rangle$  = unit binormal

(f.)  $K = \frac{1/2}{\sqrt{2}}$  = curvature

(g.)  $a = \langle -\cos t, 0, -\sin t \rangle$  = acceleration

(h.)  $a_T = \frac{0}{\sqrt{2}}$  = tangential acceleration

(i.)  $a_N = \frac{1}{\sqrt{2}}$  = centripetal acceleration

We begin by calculating the velocity

$$\mathbf{r}'(t) = \langle -\sin t, 1, \cos t \rangle$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\sin^2 t + 1 + \cos^2 t} = \boxed{\sqrt{2}} = \dot{s} \quad (\text{a.})$$

Now arclength function is defined by  $s = \int_0^t |\mathbf{r}'(u)| du$  thus

$$s = \int_0^t \sqrt{2} dt = \boxed{\sqrt{2}t} = s \quad (\text{b.})$$

The unit tangent is defined by

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \boxed{\frac{1}{\sqrt{2}} \langle -\sin t, 1, \cos t \rangle} = T(t)$$

Continuing, next the Normal vector is defined

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

so I need to calculate  $T'(t)$  &  $|T'(t)|$

$$T'(t) = \frac{d}{dt} \left[ \frac{1}{\sqrt{2}} \langle -\sin t, 1, \cos t \rangle \right] = \frac{1}{\sqrt{2}} \langle -\cos t, 0, -\sin t \rangle$$

$$|T'(t)| = \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + 0 + \sin^2 t} = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\cancel{\frac{1}{\sqrt{2}} \langle -\cos t, 0, -\sin t \rangle}}{\cancel{\sqrt{2}}}$$

$$\therefore N(t) = \langle -\cos t, 0, -\sin t \rangle \quad (\text{d.})$$

The binormal is  $B(t) = T(t) \times N(t)$ ,

$$B(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & 1 & \cos t \\ -\cos t & 0 & -\sin t \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \left( \hat{i}(-\sin t - 0) - \hat{j}(\sin^2 t + \cos^2 t) + \hat{k}(0 + \cos t) \right)$$

$$= \boxed{\frac{1}{\sqrt{2}} \langle -\sin t, -1, \cos t \rangle = B(t)} \quad (\text{e.})$$

You can check  $T \cdot B = N \cdot B = 0$  so we did the cross product w/o error.

The curvature  $\kappa$  is found from

$$\kappa = \frac{|T'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{1}{2} = \kappa} \quad (\text{f.})$$

Next, the acceleration is easy

$$\mathbf{a} = \mathbf{r}''(t) = \frac{d}{dt} \langle -\sin t, 1, \cos t \rangle = \boxed{\langle -\cos t, 0, -\sin t \rangle = \mathbf{a}} \quad (\text{g.})$$

The tangential & normal components  $a_T$  and  $a_N$  could be found directly by  $a_T = \mathbf{a} \cdot \mathbf{T}$  and  $a_N = \mathbf{a} \cdot \mathbf{N}$ , or we can use the other formulas on the sheet  $a_T = \ddot{s}$  and  $a_N = \kappa \dot{s}^2$ . I'll do both.

Sol<sup>2</sup> I

$$\begin{aligned} a_T &= \mathbf{a} \cdot \mathbf{T} = \langle -\cos t, 0, -\sin t \rangle \cdot \left( \frac{1}{\sqrt{2}} \langle -\sin t, 1, \cos t \rangle \right) \\ &= \frac{1}{\sqrt{2}} (-\sin t \cos t - \sin t \cos t) \\ &= \boxed{0 = a_T} \end{aligned}$$

$$\begin{aligned} a_N &= \mathbf{a} \cdot \mathbf{N} = \langle -\cos t, 0, -\sin t \rangle \cdot \langle -\cos t, 0, -\sin t \rangle \\ &= \cos^2 t + \sin^2 t \\ &= \boxed{1 = a_N} \end{aligned}$$

Sol<sup>2</sup> II

$$a_T = \ddot{s} = \frac{d}{dt} \left( \frac{ds}{dt} \right) = \frac{d}{dt} (\sqrt{2}) = \boxed{0 = a_T}$$

$$a_N = \kappa \dot{s}^2 = \frac{1}{2} (\sqrt{2})^2 = \boxed{1 = a_N}$$

NAME :

PROBLEM	1	2	3	4	5	6	BONUS
POINTS	25	10	10	10	40 5	5 40	5

BONUS: You may answer either (i.) or (ii.) but not both.

(i.) Calculate the torsion for  $r(t)$  in problem 5.

(ii.) Given that  $A \cdot B = A_i B_i$ ,  $A = A_i e_i$ ,  $B = B_j e_j$  and  $A \times B = \epsilon_{ijk} A_i B_j e_k$  where  $e_i \cdot e_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$  and the identity  $\epsilon_{ijk} \epsilon_{abk} = \delta_{ia} \delta_{jb} - \delta_{ja} \delta_{ib}$  prove the identity  $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ .

Show Bonus Work Here:

(i.)  $\frac{dB}{ds} = -\tau N$ . Find  $B(s)$ , note  $s = \sqrt{2}t \Rightarrow t = s/\sqrt{2}$

$$B(s) = \frac{1}{\sqrt{2}} \left\langle -\sin\left(\frac{s}{\sqrt{2}}\right), -1, \cos\left(\frac{s}{\sqrt{2}}\right) \right\rangle : \text{just substitute } \frac{s}{\sqrt{2}} \text{ for } t.$$

$$\frac{dB}{ds} = \frac{1}{2} \left\langle -\cos\left(\frac{s}{\sqrt{2}}\right), 0, -\sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle$$

$$= -\tau N(s)$$

$$= -\tau \left\langle -\cos\left(\frac{s}{\sqrt{2}}\right), 0, -\sin\left(\frac{s}{\sqrt{2}}\right) \right\rangle : \text{just put } \frac{s}{\sqrt{2}} \text{ in for } t$$

Compare these vector eq's to  
see that  $-\tau = 1/2 \Rightarrow \boxed{\tau = -1/2}$

ii) See HII problem §9.4 #30, it's exactly this.