

# MA242-011: Calculus III

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Test: #2 Form A

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**Directions:** Show your work, if you doubt that you've shown enough detail then ask.

1. (12 pts) For each of the following functions find  $f_x = \partial f / \partial x$  and  $f_y = \partial f / \partial y$ .

(a.)  $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$

$$f_x = 5x^4 + 9x^2y^2 + 3y^4$$

$$f_y = 6x^3y + 12xy^3$$

(b.)  $f(x, y) = y \ln(x)$

$$\frac{\partial f}{\partial x} = y \frac{\partial}{\partial x} [\ln(x)] = \boxed{y/x = f_x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [y \ln(x)] = \ln(x) \frac{\partial y}{\partial y} = \boxed{\ln(x) = f_y}$$

(c.)  $f(x, y) = y \sin(x^2 + y)$

$$\frac{\partial f}{\partial x} = y \frac{\partial}{\partial x} [\sin(x^2 + y)] = y \cos(x^2 + y) \frac{\partial}{\partial x} [x^2 + y]$$

$$f_x = 2xy \cos(x^2 + y^2)$$

$$\frac{\partial f}{\partial y} = \frac{\partial y}{\partial y} \sin(x^2 + y) + y \frac{\partial}{\partial y} [\sin(x^2 + y)]$$

$$= \sin(x^2 + y) + y \cos(x^2 + y) \frac{\partial}{\partial y} (x^2 + y)$$

$$= \boxed{\sin(x^2 + y) + y \cos(x^2 + y)}$$

2. (6 pts) Suppose that  $x - z = \tan^{-1}(yz)$  implicitly defines  $z$  as a function of  $x, y$  (that is  $z = z(x, y)$ ). Calculate  $\partial z / \partial y$ . Recall that  $d/du[\tan^{-1}(u)] = 1/(1+u^2)$ .

$$\frac{\partial}{\partial y}(x - z) = \frac{\partial}{\partial y}(\tan^{-1}(yz))$$

$$0 - \frac{\partial z}{\partial y} = \frac{1}{1+y^2z^2} \frac{\partial}{\partial y}(yz) = \frac{1}{1+y^2z^2}(z + y \frac{\partial z}{\partial y})$$

$$\Rightarrow \frac{-z}{1+y^2z^2} = \frac{y}{1+y^2z^2} \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \left(1 + \frac{y}{1+y^2z^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{-z}{1+y^2z^2} \left[1 + \frac{y}{1+y^2z^2}\right]^{-1}$$

$$= \frac{-z}{1+y^2z^2} \left[ \frac{1+y^2z^2+y}{1+y^2z^2} \right]^{-1}$$

$$= \frac{-z}{1+y^2z^2} \left[ \frac{1+y^2z^2}{1+y^2z^2+y} \right] = \boxed{\frac{-z}{1+y^2z^2+y} = \frac{\partial z}{\partial y}}$$

3. (10 pts) Let  $z = x \ln(x+2y)$  and suppose that  $x = \sin(t+u)$  and  $y = \cos(tu)$ . Calculate  $\partial z / \partial t$ .

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$= \left( \ln(x+2y) + \frac{x}{x+2y} \right) \frac{\partial}{\partial t}(\sin(t+u)) + \frac{\partial x}{\partial t} \frac{\partial}{\partial t}(\cos(tu))$$

(also)  
fine)  $\longrightarrow = \left[ \ln(x+2y) + \frac{x}{x+2y} \right] \cos(t+u) + \left[ \frac{2x}{x+2y} \right] (-\sin(tu) \cdot u)$

$$= \boxed{\left[ \ln(\sin(t+u) + 2\cos(tu)) + \frac{\sin(t+u)}{\sin(t+u) + 2\cos(tu)} \right] \cos(t+u) - \left[ \frac{2\sin(t+u)}{\sin(t+u) + 2\cos(tu)} \right] u \sin(tu)}$$

4. (12 pts) Let  $w = x^3 + y^3 + z^3$  and additionally constrain the variables by  $\underbrace{z = x^2 + y^3}$ . Calculate,

$$\left(\frac{\partial w}{\partial x}\right)_z \quad \left(\frac{\partial w}{\partial x}\right)_y \quad \underline{y^3 = z - x^2}$$

Remember that we must make substitutions so that only the independent variables appear before we differentiate. This is why the answers above are not equal despite the fact they are both  $\partial w / \partial x$  carelessly speaking.

$$\left(\frac{\partial w}{\partial x}\right)_z = \frac{\partial}{\partial x} \left[ x^3 + y^3 + z^3 \right] \Big|_z = \frac{\partial}{\partial x} \left[ x^3 + (z - x^2) + z^3 \right] \Big|_z = \boxed{3x^2 - 2x = \left(\frac{\partial w}{\partial x}\right)_z}$$

$$\begin{aligned} \left(\frac{\partial w}{\partial x}\right)_y &= \frac{\partial}{\partial x} \left[ x^3 + y^3 + z^3 \right] \Big|_y = \frac{\partial}{\partial x} \left[ x^3 + y^3 + (x^2 + y^3)^3 \right] \Big|_y \\ &= 3x^2 + 3(x^2 + y^3)^2 \frac{\partial}{\partial x} [x^2 + y^3] \Big|_y \\ &= \boxed{3x^2 + 6x(x^2 + y^3)^2 = \left(\frac{\partial w}{\partial x}\right)_y} \end{aligned}$$

5. (7 pts) Let us consider the level surface  $F(x, y, z) = z - e^{x^2-y^2} = 0$ . Find the normal vector at  $(1, -1, 1)$  to  $F = 0$  and write the equation of the tangent plane at that point. Recall that the normal of the tangent plane to  $F = 0$  at  $p$  is  $(\nabla F)(p)$ .

$$\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \left\langle -2xe^{x^2-y^2}, 2ye^{x^2-y^2}, 1 \right\rangle$$

$$(\nabla F)(1, -1, 1) = \langle -2, 2, 1 \rangle \quad (\text{note } e^{1-1} = e^0 = 1)$$

We have normal  $\langle -2, 2, 1 \rangle$  and  $r_0 = (1, -1, 1)$  thus

$$\boxed{-2(x-1) + 2(y+1) + 1(z-1) = 0}$$

6. (7 pts) Let us consider the parametric surface with parametrization  $X(x, y) = \langle x, y, e^{x^2-y^2} \rangle$ . Find the normal vector to this surface's tangent plane at  $X(1, -1) = (1, -1, 1)$ .

$$\begin{aligned}\boldsymbol{\Sigma}_x &= \langle 1, 0, 2xe^{x^2-y^2} \rangle \Rightarrow \boldsymbol{\Sigma}_x(1, -1) = \langle 1, 0, 2 \rangle \\ \boldsymbol{\Sigma}_y &= \langle 0, 1, -2ye^{x^2-y^2} \rangle \Rightarrow \boldsymbol{\Sigma}_y(1, -1) = \langle 0, 1, -2 \rangle\end{aligned}$$

$$N(1, -1) = (\boldsymbol{\Sigma}_x \times \boldsymbol{\Sigma}_y) \Big|_{\substack{x=1 \\ y=-1}} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix} = \boxed{\langle -2, 2, 1 \rangle}$$

7. (7 pts) Consider the graph  $z = f(x, y) = e^{x^2-y^2}$ . Find the equation for the tangent plane to the graph at  $x = 1, y = -1$ . You may recall that the tangent plane at  $(a, b)$  to  $z = f(x, y)$  was defined to be  $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$  assuming  $f$  is differentiable at  $(a, b)$ . Check that this is the same tangent plane we found in problem 5.

$$\begin{aligned}f_x &= 2xe^{x^2-y^2} \Rightarrow f_x(1, -1) = 2e^0 = 2 \\ f_y &= -2ye^{x^2-y^2} \Rightarrow f_y(1, -1) = -2e^0 = -2\end{aligned}$$

Thus, noting that  $f(1, -1) = e^{1-1} = e^0 = 1$ ,

$$\boxed{z = 1 + 2(x-1) - 2(y+1)}$$

We found  $-2(x-1) + 2(y+1) + z - 1 = 0$

This is precisely  $z = 1 + 2(x-1) - 2(y+1)$

if we rearrange the terms, hence its  
the same tangent plane.

8. (10 pts) Find the linearization of  $f(x, y, z) = \sin(xyz)$  at the point  $(1, 2, 3)$ .

$$L(x, y, z) = f(1, 2, 3) + f_x(1, 2, 3)(x-1) + f_y(1, 2, 3)(y-2) + f_z(1, 2, 3)(z-3)$$

Notice  $\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz \cos(xyz), xz \cos(xyz), xy \cos(xyz) \rangle$

So evaluating we find,

$$L(x, y, z) = \sin(6) + 6 \cos(6)(x-1) + 3 \cos(6)(y-2) + 2 \cos(6)(z-3)$$

9. (10 pts) Find the rate of change of  $f(x, y, z) = x/(y+z)$  at  $(4, 1, 1)$  in the  $\langle 1, 2, 3 \rangle$  direction.  
That is find the directional derivative of  $f$  at the point  $(4, 1, 1)$  in the  $\langle 1, 2, 3 \rangle$  direction.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{1}{y+z}, \frac{-x}{(y+z)^2}, \frac{-x}{(y+z)^2} \right\rangle$$

$$(\nabla f)(4, 1, 1) = \left\langle \frac{1}{2}, -\frac{4}{9}, -\frac{4}{9} \right\rangle = \left\langle \frac{1}{2}, -1, -1 \right\rangle$$

We need a unit vector in  $\langle 1, 2, 3 \rangle$  - direction  
this is  $\frac{1}{\sqrt{1+4+9}} \langle 1, 2, 3 \rangle = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle = \hat{u}$

Now we can compute the directional derivative,

$$\begin{aligned} (D_{\hat{u}} f)(4, 1, 1) &\equiv (\nabla f)(4, 1, 1) \cdot \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle \\ &= \left( \left\langle \frac{1}{2}, -1, -1 \right\rangle \cdot \langle 1, 2, 3 \rangle \right) \frac{1}{\sqrt{14}} \\ &= \left( \frac{1}{2} - 2 - 3 \right) \frac{1}{\sqrt{14}} \\ &= \left( \frac{1}{2} - 5 \right) \frac{1}{\sqrt{14}} \\ &= \boxed{\frac{-9}{2\sqrt{14}}} \end{aligned}$$

10. (10 pts) Assume that  $f(x, y) = e^{xy}$  has a global maximum when constrained by the condition  $x^3 + y^3 = 16$ . Find the point at which the global maximum occurs by the method of Lagrange multipliers and then find the value of the maximum.

Define  $g(x, y) = x^3 + y^3 - 16$ . Consider

$$\begin{aligned}\nabla f = \lambda \nabla g &\Rightarrow \langle ye^{xy}, xe^{xy} \rangle = \lambda \langle 3x^2, 3y^2 \rangle \\ &\Rightarrow ye^{xy} = 3\lambda x^2 \\ &xe^{xy} = 3\lambda y^2\end{aligned}$$

I.) Suppose  $x \neq 0, y \neq 0$  then can solve as follows,

$$\begin{aligned}\lambda = \frac{ye^{xy}}{3x^2} &= \frac{xe^{xy}}{3y^2} \Rightarrow \frac{y}{x^2} = \frac{x}{y^2} \Rightarrow y^3 = x^3 \\ &\Rightarrow x^3 + y^3 = 2x^3 = 16 \\ &\Rightarrow x^3 = 8 \\ &\Rightarrow \boxed{x = 2} \Rightarrow \boxed{y = 2}\end{aligned}$$

II.) Suppose  $x = 0, y = 0$  then  $0+0 \neq 16 \therefore$  violates constraint.

III.) Suppose  $x \neq 0, y = 0$  then

$$xe^{xy} = 3\lambda y^2 \Rightarrow xe^{xy} = 0 \Rightarrow x = 0, \text{ a contradiction.}$$

no such points.

IV.) Suppose  $x = 0, y \neq 0$

$$ye^{xy} = 3\lambda x^2 \Rightarrow ye^{xy} = 0 \Rightarrow y = 0 \text{ a contradiction}$$

$\therefore$  no such points.

We find that  $(2, 2)$  is where the global maximum of  $f(2, 2) = e^4$  occurs. We know this is a max since  $f(0, 0) = e^0 = 1$ .

11. (10 pts) Find the point on the plane  $x + y + z = 5$  which is closest to the point  $(1, 2, 3)$ . You may employ whichever solution you prefer, but show your work and explicitly work out any algebra that arises.

$$d^2 = (x-1)^2 + (y-2)^2 + (z-3)^2 = f(x, y, z)$$

$$g(x, y, z) = x + y + z = 5 : \text{constraint function}$$

Use Lagrange Multiplier Method,

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x-1), 2(y-2), 2(z-3) \rangle = \lambda \langle 1, 1, 1 \rangle$$

$$2(x-1) = \lambda \Rightarrow 2x = \lambda + 2$$

$$2(y-2) = \lambda \Rightarrow 2y = \lambda + 4$$

$$2(z-3) = \lambda \Rightarrow 2z = \lambda + 6$$

$$2x + 2y + 2z = 10 \quad (\quad 2g = 2(5) = 10 \quad)$$

$$\begin{matrix} \downarrow & \downarrow \\ \lambda + 2 + \lambda + 4 + \lambda + 6 & = 10 \end{matrix}$$

$$3\lambda + 12 = 10$$

$$3\lambda = 10 - 12 = -2 \quad \therefore \underline{\lambda = -2/3}$$

$$x = \frac{1}{2}(\lambda + 2) = \frac{1}{2}\left(-\frac{2}{3} + \frac{6}{3}\right) = \frac{4}{3} \cdot \frac{1}{2} = \boxed{\frac{2}{3}} = x$$

$$y = \frac{1}{2}(\lambda + 4) = \frac{1}{2}\left(-\frac{2}{3} + \frac{10}{3}\right) = \frac{10}{3} \cdot \frac{1}{2} = \boxed{\frac{5}{3}} = y$$

$$z = \frac{1}{2}(\lambda + 6) = \frac{1}{2}\left(-\frac{2}{3} + \frac{18}{3}\right) = \frac{16}{3} \cdot \frac{1}{2} = \boxed{\frac{8}{3}} = z$$

The closest point must be  $\boxed{(-2/3, 5/3, 8/3)}$

11. (10 pts) Find the point on the plane  $x + y + z = 5$  which is closest to the point  $(1, 2, 3)$ . You may employ whichever solution you prefer, but show your work and explicitly work out any algebra that arises.

Minimize  $d^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$  subject to the constraint  $x + y + z = 5$ , this time use 2<sup>nd</sup> Der. Test approach. This means I need to substitute the constraint to begin;  $f(x, y) = (x-1)^2 + (y-2)^2 + (5-y-x-3)^2 = d^2$

minimize  $f(x, y) = (x-1)^2 + (y-2)^2 + (2-y-x)^2$

$$f_x = 2(x-1) - 2(2-y-x)$$

$$f_y = 2(y-2) - 2(2-y-x)$$

Find critical point where  $\nabla f = \langle f_x, f_y \rangle = \langle 0, 0 \rangle$

$$\begin{aligned} 2x - 2 - 4 + 2y + 2x &= 4x + 2y - 6 = 0 && \text{2 eq's} \\ 2y - 2 - 4 + 2y + 2x &= 2x + 4y - 8 = 0 && \text{2 unknowns.} \end{aligned}$$

Many ways to solve, I'll use Kramer's Rule for a change,

$$\left[ \begin{array}{cc|c} 4 & 2 & x \\ 2 & 4 & y \end{array} \right] = \left[ \begin{array}{c} 6 \\ 8 \end{array} \right] \quad \rightarrow \quad x = \frac{\det \begin{bmatrix} 6 & 2 \\ 8 & 4 \end{bmatrix}}{\det \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}} = \frac{24-16}{16-8} = \frac{8}{8} = \frac{2}{3}$$

$$y = \frac{\det \begin{bmatrix} 4 & 6 \\ 2 & 8 \end{bmatrix}}{\det \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}} = \frac{32-12}{16-8} = \frac{20}{8} = \frac{5}{2}$$

Remark: you could do this algebra many ways.

Now that we've found the only critical pt.  $(\frac{2}{3}, \frac{5}{2})$  use the 2<sup>nd</sup> Derivative Test to finish,

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 16 - 4 = 12 > 0$$

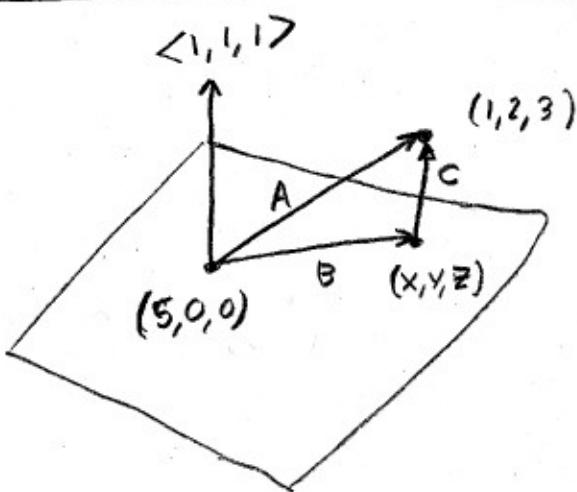
note  $f_{xx} = 4 > 0 \therefore f(1, 1)$  is a local minima and since  $f$  is differentiable everywhere this is the global minimum.

Finally the point has  $z = 5 - x - y = 5 - \frac{2}{3} - \frac{5}{2} = \frac{15-7}{3} = \frac{8}{3}$

$\therefore (\frac{2}{3}, \frac{5}{2}, \frac{8}{3})$  is the closest point

11. (10 pts) Find the point on the plane  $x + y + z = 5$  which is closest to the point  $(1, 2, 3)$ . You may employ whichever solution you prefer, but show your work and explicitly work out any algebra that arises.

Graphical Sol<sup>2</sup>: Use the vector projection to find it



① the normal of the plane is  $\langle 1, 1, 1 \rangle$ , we can read this off the eq:  $x+y+z=5$ .

② Let  $(x, y, z)$  be the point we're looking for.

③ symbolically place  $(1, 2, 3)$  in the picture.

④ it's geometrically clear that  $(1, 2, 3)$  must be in the normal direction from  $(x, y, z)$ .

⑤ let's pick a point to place the normal, let's see  $(5, 0, 0)$  will work,  $5+0+0=5$  so it's on the plane

⑥ draw some vectors  $A, B, C$  to connect what we know.

$$A = B + C$$

$$A = \langle -4, 2, 3 \rangle$$

$$B = \langle x-5, y, z \rangle$$

$$C = \langle 1-x, 2-y, 3-z \rangle$$

⑦ impose  $C \parallel \langle 1, 1, 1 \rangle$  by demanding

$$\begin{aligned} C = \text{proj}_{\langle 1, 1, 1 \rangle}(A) &= \left( \langle -4, 2, 3 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \\ &= \frac{1}{3} (-4 + 2 + 3) \langle 1, 1, 1 \rangle \\ &= \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle \end{aligned}$$

$$\text{But } C = \langle 1-x, 2-y, 3-z \rangle = \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle$$

$$\therefore 1-x = \frac{1}{3} \Rightarrow x = \frac{2}{3}$$

$$2-y = \frac{1}{3} \Rightarrow y = \frac{5}{3}$$

$$3-z = \frac{1}{3} \Rightarrow z = \frac{8}{3}$$

$$\boxed{\begin{array}{l} \left( \frac{2}{3}, \frac{5}{3}, \frac{8}{3} \right) \\ \text{closest point} \end{array}}$$

12. (5 pts) BONUS: Let  $F(x, y, z) = k$  be a level surface and suppose that  $P = (x_o, y_o, z_o)$  is a point on that surface. The tangent plane of  $F = k$  at the point  $P$  can be defined to be the union of all tangent vectors at  $P$  of curves on the level surface. If we can prove that each of these tangent vectors is perpendicular to  $(\nabla F)(P)$  this shows that  $(\nabla F)(P)$  is the normal to the tangent plane since it is perpendicular to any vector in the tangent plane. Prove that if  $r(t) = \langle f(t), g(t), h(t) \rangle$  is a curve on the level surface that has  $r(t_o) = P$  then  $(\nabla F)(P) \cdot r'(t_o) = 0$ . Note that if  $r(t)$  is on the surface then  $F(r(t)) = k$  for all  $t$ .

$$F(r(t)) = k, \quad r(t) = \langle x(t), y(t), z(t) \rangle$$

$$F(x(t), y(t), z(t)) = k$$

$$\begin{aligned} \frac{dF}{dt}(r(t)) &= \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} \\ &= (\nabla F)(r(t)) \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \frac{d}{dt}(k) \\ &= 0. \end{aligned}$$

Then when  $t = t_o$  we get  $\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \Big|_{t=t_o} = r'(t_o)$  so,

$$(\nabla F)(P) \cdot r'(t_o) = 0$$

