

TEST III SOLUTION

PROBLEM ONE

$$\begin{aligned}\int_1^2 \int_0^1 \frac{1}{(x+y)^2} dx dy &= \int_1^2 \left. \frac{-1}{x+y} \right|_0^1 dy \\ &= \int_1^2 \left(\frac{-1}{1+y} + \frac{1}{y} \right) dy \\ &= \left(-\ln|1+y| + \ln|y| \right) \Big|_1^2 \\ &= (-\ln(3) + \ln(2)) - (-\ln(2) + \ln(1)) \\ &= -\ln(3) + 2\ln(2) \\ &= \boxed{\ln(4/3)}\end{aligned}$$

PROBLEM TWO

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, x \leq z \leq 2x\}$$

$$\begin{aligned}\iiint_E yz \cos(x^5) dV &= \int_0^1 \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx \\ &= \int_0^1 \int_0^x \frac{1}{2} z^2 \Big|_x^{2x} y \cos(x^5) dy dx \\ &= \int_0^1 \int_0^x \frac{1}{2} [4x^2 - x^2] \cos(x^5) y dy dx \\ &= \int_0^1 \frac{1}{2} [3x^2] \cos(x^5) \frac{1}{2} y^2 \Big|_0^x dx \\ &= \int_0^1 \frac{3}{4} x^4 \cos(x^5) dx \\ &= \int_0^1 \frac{3}{4} \cos(u) \frac{du}{5} \\ &= \frac{3}{20} \sin(u) \Big|_0^1 \\ &= \boxed{\frac{3}{20} \sin(1)}\end{aligned}$$

$u = x^5$
 $du = 5x^4 dx$
 $x=0 \rightarrow u=0$
 $x=1 \rightarrow u=1$

PROBLEM THREE

Calculate the area of the ellipse E with boundary $x^2/a^2 + y^2/b^2 = 1$. Use $u = x/a$ & $v = y/b$.

note: $x^2/a^2 + y^2/b^2 = u^2 + v^2 = 1$

also $x = au$ $y = bv \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$

$A = \iint_E dA = \iint_{S_1} |ab| du dv$: where S_1 is the unit disk in (u,v) -space.

$= ab \iint_{S_1} du dv$

$= ab \int_0^{2\pi} \int_0^1 r dr d\theta$

$= ab \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^1 d\theta$

$= ab \frac{1}{2} \theta \Big|_0^{2\pi}$

$= \boxed{\pi ab}$

changing to polar coordinates in (u,v) -space

$u = r \cos \theta$

$v = r \sin \theta$

works same as in (x,y) -space.

not req'd
I mention
here for
complete ness.

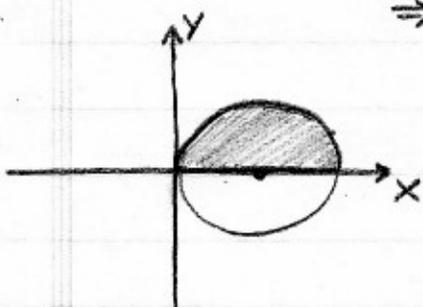
(earned a bonus pt.
if proved that
the area of unit
circle was π)

PROBLEM FOUR Considering $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} dy dx$ we can deduce $0 \leq x \leq 2$ while $0 \leq y \leq \sqrt{2x-x^2}$. We need to graph this to change to polars carefully,

$$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2$$

$$\Rightarrow x^2 - 2x + y^2 = 0$$

$$\Rightarrow (x-1)^2 + y^2 = 1 \quad ; \text{ radius one centered at } (1,0).$$



its convenient to rewrite into polars before I completed the square,

$$x^2 + y^2 = 2x$$

$$r^2 = 2r \cos \theta \quad \therefore \underline{r = 2 \cos \theta}$$

Therefore we find that this half-circle in polars is,

$$0 \leq \theta \leq \pi/2$$

$$0 \leq r \leq 2 \cos \theta$$

Now integrate,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} dy dx = \int_0^{\pi/2} \int_0^{2 \cos \theta} r \, r dr d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3} r^3 \Big|_0^{2 \cos \theta} d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3} \cos^3 \theta d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3} (1 - \sin^2 \theta) \cos \theta d\theta$$

$$= \frac{8}{3} \left(\sin \theta - \frac{1}{3} \sin^3 \theta \right) \Big|_0^{\pi/2}$$

$$= \frac{8}{3} \left(1 - \frac{1}{3} \right)$$

$$= \boxed{\frac{16}{9}}$$

$$\int (1 - \sin^2 \theta) \cos \theta d\theta$$

$$= \int (1 - u^2) du$$

$$= u - \frac{1}{3} u^3 + C$$

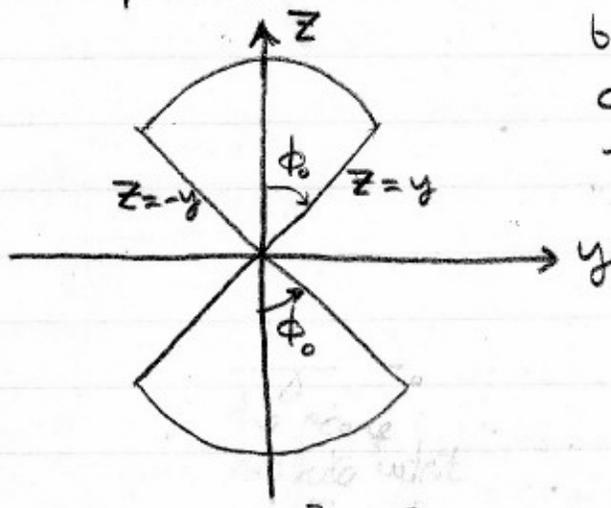
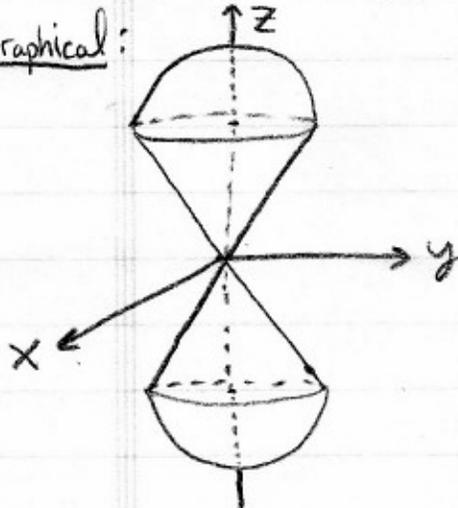
PROBLEM FIVE $x^2 + y^2 = 1$, $z = x + y + 1$, $z = 0$ is
 simple in cylindricals: $r^2 = 1$, $z = r \cos \theta + r \sin \theta + 1$, $z = 0$
 which indicates $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$, $0 \leq z \leq r(\cos \theta + \sin \theta)$.

$$\begin{aligned}
 \iiint_E x \, dV &= \int_0^{2\pi} \int_0^1 \int_0^{r(\cos \theta + \sin \theta)} r \cos \theta \, r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 r^2 \cos \theta [r(\cos \theta + \sin \theta)] \, dr \, d\theta \\
 &= \int_0^{2\pi} (\cos^2 \theta + \sin \theta \cos \theta) \, d\theta \int_0^1 r^3 \, dr \\
 &= \int_0^{2\pi} \left[\frac{1}{2}(1 + \cos 2\theta) + \sin \theta \cos \theta \right] \, d\theta \cdot \frac{r^4}{4} \Big|_0^1 \\
 &= \left(\frac{\theta}{2} + \frac{1}{4} \sin(2\theta) + \frac{1}{2} \sin^2 \theta \right) \Big|_0^{2\pi} \cdot \frac{1}{4} \\
 &= \boxed{\pi/4}
 \end{aligned}$$

PROBLEM SIX

Find volume bounded by $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = 1$. (this is $\rho^2 = 1$)

(I) Graphical:



by symmetry can look at the $x=0$ cross-section to figure out the angle ϕ_0 the cone makes w.r.t z -axis.

$$x=0 \begin{cases} z^2 = y^2 \Rightarrow z = \pm y \\ y^2 + z^2 = 1 \end{cases}$$

$z = y$ makes a 45° angle ϕ_0 or $\phi_0 = \pi/4$ by graph.

(II) Algebraic method to find ϕ_0

$$\begin{aligned} x &= \rho \cos \theta \sin \phi \\ y &= \rho \sin \theta \sin \phi \\ z &= \rho \cos \phi \end{aligned} \Rightarrow \begin{cases} x^2 + y^2 = \rho^2 \sin^2 \phi \Rightarrow \sqrt{x^2 + y^2} = \rho \sin \phi \\ z = \rho \cos \phi \Rightarrow \tan \phi = \frac{z}{\sqrt{x^2 + y^2}} \end{cases}$$

now apply the cone eqⁿ

$$\tan \phi_0 = \frac{z}{\sqrt{x^2 + y^2}} = \frac{z}{\sqrt{z^2}} = \pm 1 \Rightarrow \phi_0 = \frac{\pi}{4}, \frac{3\pi}{4}$$

So however you like to see it we have,

$$0 \leq \rho \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi/4 \text{ and } \frac{3\pi}{4} \leq \phi \leq \pi$$

alternatively could use $0 \leq \phi \leq \pi/4$ then double it by symmetry.

Thus

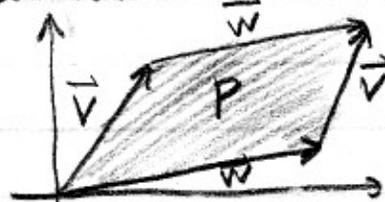
$$\begin{aligned} V &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/4} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho + \int_0^1 \int_0^{2\pi} \int_{3\pi/4}^{\pi} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\ &= \left(\frac{1}{3}\right)(2\pi) \left(-\cos \phi \Big|_0^{\pi/4}\right) + \left(\frac{1}{3}\right)(2\pi) \left(-\cos \phi \Big|_{3\pi/4}^{\pi}\right) \\ &= \frac{2\pi}{3} \left(-\frac{\sqrt{2}}{2} + 1\right) + \frac{2\pi}{3} \left(1 - \cos(3\pi/4)\right) \\ &= \boxed{\frac{4\pi}{3} \left[1 - \frac{\sqrt{2}}{2}\right]} \end{aligned}$$

A possible solⁿ to the bonus question

Our goal: $T(u, v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc \neq 0$ takes parallelogram P in (u, v) -space to a new parallelogram $T(P)$ in (x, y) -space. Show that

$$\boxed{\text{Area}(T(P)) = |\det(A)| \text{Area}(P)} \quad (\text{Prove this.})$$

For convenience we can consider a P at the origin



$$\text{Area}(P) = |\vec{v} \times \vec{w}|$$

Now $\vec{v} = (v_1, v_2, 0)$ and $\vec{w} = (w_1, w_2, 0)$ we can calculate $\vec{v} \times \vec{w} = (0, 0, v_1 w_2 - v_2 w_1)$ thus $|\vec{v} \times \vec{w}| = |v_1 w_2 - v_2 w_1| = \text{Area}(P)$.

$$T(\vec{v}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$$

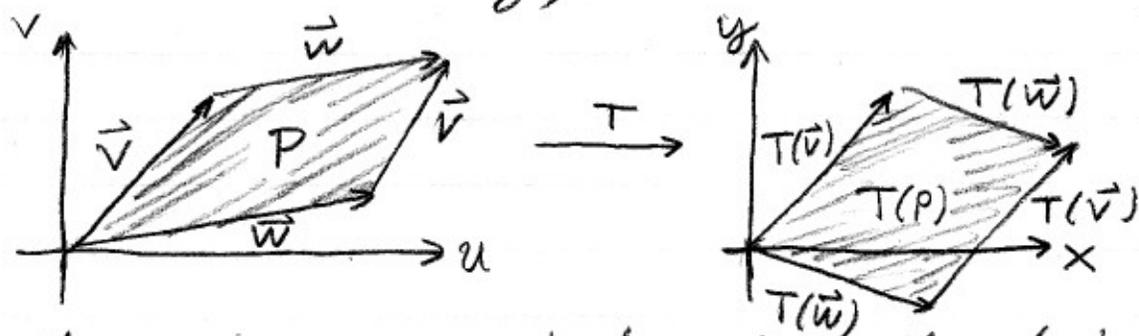
$$T(\vec{w}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} aw_1 + bw_2 \\ cw_1 + dw_2 \end{bmatrix}$$

$$\begin{aligned} T(\vec{v}) \times T(\vec{w}) &= (av_1 + bv_2, cv_1 + dv_2, 0) \times (aw_1 + bw_2, cw_1 + dw_2, 0) \\ &= (0, 0, (av_1 + bv_2)(cw_1 + dw_2) - (cv_1 + dv_2)(aw_1 + bw_2)) \end{aligned}$$

$$\begin{aligned} & \cancel{acv_1w_1} + \cancel{ad} \underline{v_1w_2} + \cancel{bc} \underline{v_2w_1} + \cancel{bd} \underline{v_2w_2} - \cancel{cav_1w_1} - \cancel{cb} \underline{v_1w_2} - \cancel{d} \underline{a} \underline{v_2w_1} - \cancel{d} \underline{b} \underline{v_2w_2} \\ & (ad - bc) \underline{v_1w_2} - (ad - bc) \underline{v_2w_1} \\ & (ad - bc) (v_1w_2 - v_2w_1) \\ & \det(A) (\vec{v} \times \vec{w})_3 \end{aligned}$$

$$\therefore \underline{T(\vec{v}) \times T(\vec{w}) = (0, 0, \det(A) (v_1w_2 - v_2w_1))}$$

Let me draw a picture of what we are calculating,



Notice then using our algebra from the last page we find,

$$\begin{aligned}
 \text{Area}(T(P)) &= |T(\vec{v}) \times T(\vec{w})| \\
 &= |(0, 0, \det(A)(v_1 w_2 - v_2 w_1))| \\
 &= \sqrt{[\det(A)(v_1 w_2 - v_2 w_1)]^2} \\
 &= \sqrt{[\det(A)]^2} \sqrt{(v_1 w_2 - v_2 w_1)^2} \\
 &= |\det(A)| |v_1 w_2 - v_2 w_1| \\
 &= |\det(A)| \text{Area}(P).
 \end{aligned}$$

This result is at the heart of the coordinate change $\mathbb{R}^n \rightarrow \mathbb{R}^n$, it's the reason the det of the derivative matrix shows up. In short you can approximate $T(u, v)$ locally by a linear function whose matrix is the jacobian matrix, generally T is not linear but infinitesimally T is linear. You can read my notes for a better discussion or better yet read Colley.