Solution to Test III, Laplace Transforms

PROBLEM ONE Suppose $w(1) = 1$ and $w'(1) = 0$. Solve

$$w''(t) - w(t) = \sin(t-1)$$

Using Laplace transforms. Introduce $y(t) = w(t+1)$ so that $y(0) = w(1)$. Also notice that
$y'(t) = \frac{dw}{dt}\frac{d}{dt}(t+1) = w'(t+1)$ and $y''(t) = w''(t+1)$
Consider the differential equation at $t+1$,

$$w''(t+1) - w(t+1) = \sin(t+1-1)$$

Thus, $y''(t) - y(t) = \sin(t)$, and $y(0) = 1$, $y'(0) = 0$.
Now we can use the standard Laplace theory on y,

$$s^2 Y - s - Y = \frac{1}{s^2+1}$$

$$Y = \frac{1}{s^2-1}\left(s + \frac{1}{s^2+1}\right)$$

$$= \frac{s(s^2+1)+1}{(s+1)(s-1)(s^2+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C+Ds}{s^2+1}$$

$$\Rightarrow \quad s(s^2+1)+1 = A(s-1)(s^2+1) + B(s+1)(s^2+1) + (s^2-1)(C+Ds)$$

$$\Rightarrow \quad s^3+s+1 = A(s^3-s^2+s-1) + B(s^3+s^2+s+1) + C(s^2-1) + D(s^3-s)$$

Equating Coefficients of $s^3, s^2, s, 1$ yields,

s^3 | $1 = A + B + D$

s^2 | $0 = -A + B + C$

s | $1 = A + B - D$

1 | $1 = -A + B - C$

Well that's not very fun, notice $s = 1$ gives $3 = 4B$
whereas $s = -1$ gives $-1 = -4A$

PROBLEM ONE Continued

Just found $A = 1/4$ and $B = 3/4$ so using the eq^n's from last page,

$$1 = A + B + D = 1 + D \quad \therefore \quad \underline{D = 0}$$

$$0 = -A + B + C = \frac{1}{2} + C \quad \therefore \quad \underline{C = -1/2}$$

Thus we find that

$$Y = \frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{3}{4}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

Thus $y(t) = \mathcal{L}^{-1}\{Y\}(t)$ is clearly,

$$y(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{t} - \frac{1}{2}\sin(t)$$

Now to finish the problem we convert back to w using the fact $w(t) = y(t-1)$ thus,

$$\boxed{w(t) = \frac{1}{4}e^{-(t-1)} + \frac{3}{4}e^{t-1} - \frac{1}{2}\sin(t-1)}$$

There are other perhaps simpler ways to express our final answer, but this will suffice.

PROBLEM TWO

Solve $x' = -x + 2y + u(t-3)$

$y' = -2y + x$

with $x(0) = 1$ and $y(0) = 0$. Notice we would have difficulty treating $u(t-3)$ w/o the Laplace transform technique. Anyway, lets begin,

$$s\mathbb{X} - 1 = -\mathbb{X} + 2\mathbb{Y} + \frac{1}{s}e^{-3s}$$
$$s\mathbb{Y} = -2\mathbb{Y} + \mathbb{X}$$

We can clean these up a little,

$$(s+1)\mathbb{X} - 2\mathbb{Y} = 1 + \frac{1}{s}e^{-3s}$$
$$(s+2)\mathbb{Y} = \mathbb{X}$$

Substitute the 2nd Eq^{n} into the first,

$$(s+1)(s+2)\mathbb{Y} - 2\mathbb{Y} = 1 + \frac{1}{s}e^{-3s}$$
$$(s^2 + 3s)\mathbb{Y} = 1 + \frac{1}{s}e^{-3s}$$
$$\underbrace{\mathbb{Y} = \frac{1}{s(s+3)} + \frac{1}{s^2(s+3)}e^{-3s}}_{\text{I}}$$

Now we can see that,

$$\underbrace{\mathbb{X} = \frac{s+2}{s(s+3)} + \frac{s+2}{s^2(s+3)}e^{-3s}}_{\text{II}}$$

Then the problem reduces to finding the inverse Laplace transforms of I & II. This will be accomplished with the help of four partial fractal decompositions.

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \Rightarrow 1 = A(s+3) + Bs$$

$s=0 \Rightarrow 1 = 3A \therefore A = 1/3$

$s=-3 \Rightarrow 1 = -3B \therefore B = -1/3$

$$\underbrace{\frac{1}{s(s+3)} = \frac{1}{3s} - \frac{1}{3}\left(\frac{1}{s+3}\right)}_{(i)}$$

$$F(s) = \frac{1}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$1 = As(s+3) + B(s+3) + Cs^2$

$s=0$: $1 = 3B \rightarrow B = 1/3$

$s=-3$: $1 = 9C \rightarrow C = 1/9$

$s=1$: $1 = 4A + 4B + C$

$$A = \frac{1}{4}(1-4B-C) = \frac{1}{4}\left(1 - \frac{4}{3} - \frac{1}{9}\right) = \frac{1}{4}\left(\frac{9-12-1}{9}\right) = \frac{-1}{9}$$

$$F(s) = \frac{-1}{9s} + \frac{1}{3s^2} + \frac{1}{9}\left(\frac{1}{s+3}\right)$$

$$\underbrace{f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{-1}{9} + \frac{t}{3} + \frac{1}{9}e^{-3t}}_{(ii)}$$

So now we have the needed ingredients to compute $\mathcal{L}^{-1}\{Y\}$ (I)

$$Y(t) = \mathcal{L}^{-1}\{\overbrace{\frac{1}{3s} - \frac{1}{3}\left(\frac{1}{s+3}\right)}^{\text{from (i)}}\}(t) + \mathcal{L}^{-1}\{F(s)e^{-3s}\}(t)$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3t} + f(t-3)\,u(t-3)$$

$$= \boxed{\frac{1}{3} - \frac{1}{3}e^{-3t} + \underbrace{\left(\frac{-1}{9} + \frac{1}{3}(t-3) + \frac{1}{9}e^{-3(t-3)}\right)}_{\text{Used (ii)}}u(t-3) = Y(t)}$$

PROBLEM TWO CONTINUED

$$\frac{s+2}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \longrightarrow s+2 = A(s+3) + Bs$$

$$\underline{s=0|}\quad 2 = 3A \quad \therefore\ A = 2/3$$

$$\underline{s=-3|}\quad -1 = -3B \quad \therefore\ B = 1/3$$

$$\underline{\frac{s+2}{s(s+3)} = \frac{2}{3s} + \frac{1}{3}\left(\frac{1}{s+3}\right)} \quad (iii)$$

$$G(s) = \frac{s+2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s+2 = As(s+3) + B(s+3) + Cs^2 = s^2(A+C) + s(3A+B) + 3B$$

$$\underline{s^2|}\quad 0 = A+C \quad \rightarrow C = -A$$

$$\underline{s|}\quad 1 = 3A+B$$

$$\underline{1|}\quad 2 = 3B \rightarrow B = 2/3 \longrightarrow A = (1-B)/3 = 1/9 \rightarrow C = -1/9.$$

$$G(s) = \frac{1}{9s} + \frac{2}{3s^2} - \frac{1}{9(s+3)}$$

$$\underline{g(t) = \mathcal{L}^{-1}\{G\}(t) = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t} = g(t)} \quad (iv)$$

Finally compute $x(t) = \mathcal{L}^{-1}\{X\}(t)$ from II using (iii) & (iv),

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2}{3s} + \frac{1}{3(s+3)}\right\}(t) + \mathcal{L}^{-1}\{G(s)e^{-3s}\}(t)$$

$$x(t) = \frac{2}{3} + \frac{1}{3}e^{-3t} + g(t-3)u(t-3)$$

$$\boxed{x(t) = \frac{2}{3} + \frac{1}{3}e^{-3t} + \left(\frac{1}{9} + \frac{2}{3}(t-3) - \frac{1}{9}e^{-3(t-3)}\right)u(t-3)}$$