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Practice Test II: Systems of ODEs

WARNING: this is a practice test, I will make the actual test shorter. Some problems will remain essentially the same but there is obviously too much for 75 minutes here. I'll begin with the possible proof problems, I will give the formula for x_p if it is needed, however I would like for you to remember the formulas for the real solutions contained within the complex solution. I also expect you to understand how to solve $\vec{x}' = A\vec{x}$ using the matrix exponential, I will give you the formula

$$e^{At} = e^{\lambda t}(I + t(A - \lambda I) + \frac{1}{2}t^2(A - \lambda I)^2 + \frac{1}{3!}t^3(A - \lambda I)^3 + \cdots)$$

however, I expect you to know what to do with it. The format of the test will be three problems like the 29pt problems, 1 of the proof problems just as they are stated here, and two more challanging 4pt problems. There will be 105pts weighted by 100, that is 5 bonus pts possible. It is especially important that you explain steps on this test.

1. (10pts) Show that $\vec{x} = e^{\lambda t}\vec{u}$ is a nonzero solution to $\vec{x}' = A\vec{x}$ if we require that λ and $\vec{u}$ are constant with $det(A - \lambda I) = 0$ and $(A - \lambda I)\vec{u} = 0$. You will need to use a theorem from linear algebra which states that $B\vec{u} = 0$ has more than one solution only if $det(B) = 0$.

2. (10pts) Show that if $\vec{w} = Re(\vec{w}) + iIm(\vec{w})$ is a solution to $\vec{w}' = A\vec{w}$ then both $Re(\vec{w})$ and $Im(\vec{w})$ are also solutions.

3. (10pts) Show that if $\lambda = \alpha + i\beta$ and $\vec{u} = \vec{a} + i\vec{b}$ then

$$Re(e^{\lambda t}\vec{u}) = e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b} \qquad and \qquad Im(e^{\lambda t}\vec{u}) = e^{\alpha t}\sin(\beta t)\vec{a} + e^{\alpha t}\cos(\beta t)\vec{b}.$$

Notice that we then have found how to extract two real solutions from the complex solution. I should mention that I assume here that $\alpha, \beta, \vec{a}, \vec{b}$ are all real, they have no $i = \sqrt{-1}$.

4. (10 pts) Show that $\vec{x_p} = X\vec{v}$ is a solution to $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ if,

$$\vec{x_p}(t) = X(t)\int X^{-1}(t)\vec{f}(t)dt.$$

Here we assume X is a fundamental matrix for the system.

5. (10pts) Show that the matrix exponential is a fundamental matrix. That is show that e^{At} is invertible and it is a solution matrix for $\vec{x}' = A\vec{x}$.

6. (29pts) Rewrite the following system of differential equations in matrix normal form

$$x' = 2x - y \qquad\qquad y' = x + 2y.$$

Now find the general solution using our eigenvalue/eigenvector technique. Finally find the solution with $x(0) = 0$ and $y(0) = 1$ and write out the formulas for $x(t)$ and $y(t)$ seperately.

7. (29pts) Find the eigenvalues and generalized eigenvectors of the matrix below

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

8. (29pts) Find the eigenvalues and generalized eigenvectors of the matrix below

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}$$

then find the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$.

9. (29pts) Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$$

10. (29pts) Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

11. (29pts) Find the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

12. (29pts) Solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ given that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

13. (29pts) Suppose that A is a 5x5 matrix such that $det(A-\lambda I) = (\lambda-1)^2(\lambda-3)^3$. Furthermore, suppose that

$$(A - I)\vec{u_1} = 0 \qquad and \qquad (A - I)\vec{u_2} = 0$$

where $\vec{u_1}, \vec{u_2}$ are nontrivial and linearly independent. Next suppose that,

$$(A - 3I)\vec{u_3} = 0 \qquad and \qquad (A - 3I)\vec{u_4} = \vec{u_3} \qquad and \qquad (A - 3I)\vec{u_5} = \vec{u_4}$$

where $\vec{u_3}, \vec{u_4}, \vec{u_5}$ are all nontrivial. Given all this data calculate the general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors. You may use the formula on the board without proof. However, you should certainly show your work.