— SOLUTION —

YOUR NAME HERE:

MA 341, Introduction to Differential Equations

Instructor: James Cook
Test I: n-th order ODEs
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Directions: Show your work, if you doubt that you've shown enough detail then ask. If you need additional paper please ask. There are 106 pts to be earned, 6pts are bonus.

1. (10pts) Given that $v(0) = 10$ solve

$$\frac{dv}{dt} = -v$$

If v represents the velocity of a cat thrown horizontally with an initial velocity of 10 then does the cat ever come to rest ? Here you can interpret the term $-v$ as a frictional force. For a **bonus point find if the position of the cat is bounded, use $v = dx/dt$.**

$$\frac{dV}{dt} = -V \Rightarrow \int \frac{dV}{V} = \int -dt$$

$$\Rightarrow \ln|V| = -t + C$$

$$\Rightarrow V = \pm e^c e^{-t} = C_2 e^{-t}$$

$$V(0) = 10 = C_2 \quad \therefore \quad \boxed{V = 10e^{-t}}$$

- The cat never comes to rest since $V \neq 0$ for any t. ($V \longrightarrow 0$ as $t \longrightarrow \infty$ but $t \neq \infty$)

- $X(t) - X_0 = \int_0^t \frac{dx}{du} du$ — (Bonus) —

$$= \int_0^t 10e^{-u} du$$

$$= -10e^{-u} \Big|_0^t$$

$$= -10e^{-t} + 10$$

$$\therefore \quad X(t) - X_0 = 10(1 - e^{-t})$$

as $t \to \infty$ we see $X(t) - X_0 \longrightarrow 10$ so the motion is bounded. Notice that the cat does not reach $X = X_0 + 10$ in <u>finite</u> time.

2. (11pts) Assume that $x > 0$ for this problem. Solve

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1$$

$$\frac{dy}{dx} - \frac{1}{x} y = 2x + 1$$

$$\mu = \exp\left(\int \frac{-1}{x} dx\right) = \exp(-\ln|x|) = \exp\left(\ln\left(\frac{1}{x}\right)\right) = \frac{1}{x}.$$

Multiply by μ.

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = \frac{2x+1}{x} = 2 + \frac{1}{x}$$

Use product rule,

$$\frac{d}{dx}\left(\frac{1}{x} y\right) = 2 + \frac{1}{x}$$

integrate both sides,

$$\int \frac{d}{dx}\left(\frac{1}{x} y\right) dx = \frac{1}{x} y = \int \left(2 + \frac{1}{x}\right) dx = 2x + \ln|x| + C$$

$$\therefore \boxed{y = 2x^2 + x\ln(x) + Cx}$$

(We used $|x| = x$ since $x > 0$.)

3. (10 pts) Find the solution of

$$2xdx + 2ydy = 0$$

that passes through the point $(1, 1)$. Notice this is an exact equation.

$$\frac{\partial F}{\partial x} = 2x \quad \& \quad \frac{\partial F}{\partial y} = 2y$$

$$F = \int \frac{\partial F}{\partial x}\, dx = \int 2x\, dx = x^2 + C_1(y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x^2 + C_1) = \frac{\partial C_1}{\partial y} = 2y$$

$$\therefore \ C_1 = \int 2y\, dy = y^2 + C_2$$

Thus $F = x^2 + y^2 + C_2$ and

sol^{n}'s have the form

$$x^2 + y^2 = k$$

(circles for $k \neq 0$)

$$x = y = 1 \implies 1^2 + 1^2 = k = 2$$

$$\therefore \ \boxed{x^2 + y^2 = 2}$$

4. (24pts) Find the general solutions to the following differential equations, as usual we denote $D = d/dx$ and $y' = dy/dx$ etc...

(a.) $y'' + 6y' + 9y = 0$

(b.) $y'' - y = 0$

(c.) $(D^3 - 10D^2 + 26D)[y] = 0$

(d.) $(D^2 + 1)^2[y] = 0$

(a.) $\lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0 \quad \therefore \lambda_1 = -3, \lambda_2 = -3$

$$\boxed{y = c_1 e^{-3x} + c_2 x e^{-3x}}$$

(b.) $\lambda^2 - 1 = 0 \to \lambda^2 = 1 \to \lambda = \pm 1$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x}}$$

(c.) $\lambda^3 - 10\lambda^2 + 26\lambda = 0$

$\lambda(\lambda^2 - 10\lambda + 26) = 0$

$\lambda_1 = 0, \quad \lambda_{2,3} = \dfrac{10 \pm \sqrt{100-104}}{2} = 5 \pm i$

$$\boxed{y = c_1 + c_2 e^{5x}\cos(x) + e^{5x}\sin(x)}$$

(d.) $(\lambda^2+1)^2 = 0 \to \lambda = \pm i$ <u>twice</u>

$$\boxed{y = c_1\cos(x) + c_2\sin(x) + c_3 x\cos(x) + c_4 x\sin(x)}$$

(5.) (25pts) Find the general solution of

$$y'' = 6x + e^x$$

$\lambda^2 = 0 \rightarrow y_h = c_1 + c_2 x$

Choose $A = D^2(D-1)$ then $A[6x+e^x] = 0$ as desired. Now write our original eq$^{\underline{n}}$ as

$$D^2[y] = 6x + e^x$$

operate by A on both sides,

$$D^2(D-1)D^2[y] = 0$$

$$\lambda^2(\lambda-1)\lambda^2 = 0$$

$$\hookrightarrow y = \underbrace{c_1 + c_2 x}_{y_h} + \underbrace{c_3 e^x + c_4 x^2 + c_5 x^3}$$

$$y_p = Ae^x + Bx^2 + Cx^3$$

$$y_p' = Ae^x + 2Bx + 3Cx^2$$

$$y_p'' = Ae^x + 2B + 6Cx$$

$$y_p'' = Ae^x + 2B + 6Cx = 6x + e^x$$

$$\Rightarrow A = 1, \quad 2B = 0 \quad \& \quad 6C = 6$$

$$\therefore \quad \underline{A = 1, \quad B = 0, \quad C = 1}$$

Therefore the general sol$^{\underline{n}}$ follows,

$$\boxed{y = c_1 + c_2 x + e^x + x^3}$$

6 ~~1~~. (10 pts) Use the method of anihilators to find the correct form for the particular solution (don't find $A, B, C, \ldots$ just set it up) of the following differential equation, $D = d/dx$ as usual,

$$(D+1)(D^2+1)(D-3)^2[y](x) = e^{-x}$$

Here $A = D+1$ will have $A[e^{-x}] = 0$.

then we get,

$$(D+1)^2(D^2+1)(D-3)^2[y] = 0$$

which has sol^{n}'s,

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 \cos(x) + c_4 \sin(x) + c_5 e^{3x} + c_6 x e^{3x}$$

then we can see the $c_2 x e^{-x}$ term is the one that doesn't appear in y_h. Hence,

$$\boxed{y_p = Axe^{-x}}$$

7. (5pts) Use $e^{ix} = cos(x) + isin(x)$ to show that

$$sin(2x) = 2sin(x)cos(x)$$

it is useful to first recall the formula for cos(x) and sin(x) in terms of e^{ix}.

You could have either recalled or derived that

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \& \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Let me derive them for the sake of completeness,

$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos(x) - i\sin x$$

Thus

$$e^{ix} + e^{-ix} = 2\cos(x) \Rightarrow \cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$
$$e^{ix} - e^{-ix} = 2i\sin(x) \Rightarrow \sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$

What follows was the essential portion for credit,

$$2\sin(x)\cos(x) = \frac{2}{2i}(e^{ix} - e^{-ix})\frac{1}{2}(e^{ix} + e^{-ix})$$
$$= \frac{1}{2i}(e^{2ix} + 1 - 1 - e^{-2ix})$$
$$= \frac{1}{2i}(e^{2ix} - e^{-2ix})$$
$$= \sin(2x).$$

8. (10pts) Let us return to the problem of integration, given some h our goal was to find y such that

$$\int h dx = y$$

where we required that y is the antiderivative of h, that is

$$y' = h.$$

With the above in mind, solve the following integral via undetermined coefficients.

$$\int x^2 e^x dx$$

(do not use integration by parts, think outside the box)

Solve $\dfrac{dy}{dx} = x^2 e^x$

we have $\lambda = 0 \therefore y_h = c_1$.

Next choose $A = (D-1)^3$ to get $A[x^2e^x] = 0$.

then $D[y] = x^2 e^x$ converts to

$(D-1)^3 D[y] = 0$

$\Rightarrow \underbrace{y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x}_{y_p = Ae^x + Bxe^x + Cx^2e^x} + \underbrace{c_4}_{y_h}$

$y_p = e^x(A + Bx + Cx^2)$

$y_p' = e^x(A+Bx+Cx^2) + e^x(B + 2Cx)$

$y_p' = e^x(A + Bx + Cx^2) + e^x(B + 2Cx) = x^2 e^x$

$\underline{e^x}|$ $A + B = 0$

$\underline{xe^x}|$ $B + 2C = 0$

$\underline{x^2e^x}|$ $C = 1 \rightarrow B = -1 \rightarrow A = 1$

$\therefore$ $\boxed{y = e^x - xe^x + x^2e^x + c_1 = \int x^2 e^x dx}$