Non Homogeneous Linear Systems

We wish to solve (A constant and $f(t)$ continous)

$$\boxed{x'(t) = Ax(t) + f(t)} \quad Eq^n (1)$$

Undetermined "Coefficient"

If $f(t)$ is made of polynomials, exponentials and sines and cosines then we can guess x_p has a similar form just like in n^{th} order ODE theory.

$\boxed{E1}$ $x'(t) = Ax(t) + tg$ where $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ & $g = \begin{bmatrix} -9 \\ 0 \\ 18 \end{bmatrix}$

It can be shown that the corresponding homogeneous eq^n $x' = Ax$ has the sol^n

$$x_h = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Then a good guess for x_p is

$$x_p = ta + b$$
$$x_p' = a$$

Substituting x_p into $Eq^n(1)$ yields,

$$a = A(ta+b) + tg = t(Aa + g) + Ab = a$$

Equating coeff. of t and 1 yields

$$Ab = a$$
$$Aa + g = 0$$

Now $\det(A) = -27$ so infact A^{-1} exists, we can calculate that $A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ thus

$$Aa + g = 0 \Rightarrow a = -A^{-1}g = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = a$$

Then $Ab = a = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \Rightarrow b = A^{-1}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = b$

$\boxed{E1}$ Conclusion, we find general sol$^{\underline{n}}$

$$X = X_h + X_p$$

$$\boxed{X = c_1 e^{3t}\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}-1\\1\\0\end{bmatrix} + c_3 e^{-3t}\begin{bmatrix}-1\\-1\\1\end{bmatrix} + t\begin{bmatrix}3\\6\\0\end{bmatrix} + \begin{bmatrix}-1\\0\\2\end{bmatrix}}$$

Notice the text had $g = [-9 \;\; 0 \; -18]$ whereas I took 18 instead. It's interesting how much a single sign will modify results.

<u>Remark</u>: this method is not very good, it leads us to algebra which is often not well posed. I mean you really need to be a bit sneaky to solve it. The next method will require less sneakyness.

VARIATION OF PARAMETERS FOR SYSTEMS

Again we wish to solve $x'(t) = A(t)x(t) + f(t)$. Notice we do not necessarily demand $A(t)$ is a constant matrix. Note if we assume $A(t), f(t)$ are continous on I then we know the sol$^{\underline{n}}$ has the form

$$x(t) = \Phi c + x_p$$

where Φ is the fundamental matrix $\Phi = (x_1|x_2|\cdots|x_n)$ which has $x_1, x_2, \dots, x_n$ – LI sol$^{\underline{n}}$'s to $x' = Ax$. Now suppose we have found Φ somehow, we guess that the particular sol$^{\underline{n}}$ has the form

$$x_p = \Phi v$$

where $v = [v_1, v_2, \dots, v_n]^T$ is an vector of unknown functions which we wish to determine. Calculate then

$$\frac{dx_p}{dt} = \frac{d\Phi}{dt}v + \Phi\frac{dv}{dt}$$

Substitute the above into the eq$^{\underline{n}}$: $\dfrac{dx_p}{dt} = Ax_p + f$

$$\frac{d\Phi}{dt}v + \Phi\frac{dv}{dt} = A\Phi v + f$$

Now if you think about it $\Phi' = A\Phi$ thus

$$\frac{d\Phi}{dt}v + \Phi\frac{dv}{dt} = \frac{d\Phi}{dt}v + f$$

$$\therefore \quad \Phi\frac{dv}{dt} = f \quad \Rightarrow \quad \boxed{\frac{dv}{dt} = \Phi^{-1}f} \qquad \text{Eq}^{\underline{n}}\ (\star)$$

Now we integrate to obtain

$$v(t) = \int \Phi^{-1}(t)f(t)\,dt$$

Yielding the particular sol$^{\underline{n}}$

$$\boxed{x_p = \Phi\int\Phi^{-1}f\,dt}$$

Given the x_p we found the general sol^{n} is

$$\boxed{x = \Phi c + \Phi \int \Phi^{-1} f \, dt} \qquad \text{Eq}^{n}(11)$$

Further suppose that $x(t_o) = x_o$ is given then following our previous derivation from Eq^{n}(~~?~~) we integrate from t_o to t (as opposed to the indefinite integral from before)

$$x(t) = \Phi(t)c + \Phi(t)\int_{t_o}^{t} \Phi^{-1}(u) f(u)\,du$$

Then $x(t_o)$ yields an $\int_{t_o}^{t_o}$ which vanishes giving

$$x(t_o) = \Phi(t_o)c = x_o \quad \Rightarrow \quad c = \Phi^{-1}(t_o)\, x_o$$

Thus, the following satisfies $x' = Ax + f$ with $x(t_o) = x_o$

$$\boxed{x(t) = \Phi(t)\, \Phi^{-1}(t_o)\, x_o + \Phi(t)\int_{t_o}^{t} \Phi^{-1}(u) f(u)\,du} \qquad \text{Eq}^{n}(13)$$

Notice here the t verses t_o dependence is important to indicate, I cannot just drop the t-dependence w/o danger of confusion. Anyway Eq^{n}(13) is remarkable, compared with undetermined "coeff" it is much more straight forward to calculate. Trouble is we need to find Φ. We seen how to do that for a special case

$$\frac{dx}{dt} = Ax$$

where $A'(t) = 0$ and A has n-LI eigenvectors. But, what if $\nexists$ enough eigenvectors for A? We answer that question next. Another obvious question is what about when $A'(t) \neq 0$ what do we do then? I have no general answer to that, just like in the n^{th} order theory we could resort to Laplace transforms, but there's no gaurantee that the inverse transforms will be tractable.