Problem 42

$*dy = -(dz \wedge dx \wedge dt)$

$$\begin{aligned}
*dy &= \frac{1}{1!}\frac{1}{(4-1)!}\delta^{\mu 2}\epsilon_{\mu\nu\alpha\beta}dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta} \\
&= \frac{1}{6}\left[\epsilon_{2\nu\alpha\beta}(dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta})\right] \\
&= \frac{1}{6}[\overbrace{\epsilon_{2013}}^{1}\overbrace{(dt\wedge dx\wedge dz)}^{-1}+\overbrace{\epsilon_{2031}}^{-1}\overbrace{(dt\wedge dz\wedge dx)}^{1}+\overbrace{\epsilon_{2103}}^{-1}\overbrace{(dx\wedge dt\wedge dz)}^{1} \\
&\quad +\overbrace{\epsilon_{2130}}^{1}\overbrace{(dx\wedge dz\wedge dt)}^{-1}+\overbrace{\epsilon_{2301}}^{1}\overbrace{(dz\wedge dt\wedge dx)}^{-1}+\overbrace{\epsilon_{2310}}^{-1}\overbrace{(dz\wedge dx\wedge dt)}^{1}] \\
&= \frac{1}{6}[-6(dz\wedge dx\wedge dt)] \\
&= -dz\wedge dx\wedge dt
\end{aligned}$$

$*dz = -(dx \wedge dy \wedge dt)$

$$\begin{aligned}
*dz &= \frac{1}{1!}\frac{1}{(4-1)!}\delta^{\mu 3}\epsilon_{\mu\nu\alpha\beta}dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta} \\
&= \frac{1}{6}\left[\epsilon_{3\nu\alpha\beta}(dx^{\nu}\wedge dx^{\alpha}\wedge dx^{\beta})\right] \\
&= \frac{1}{6}[\overbrace{\epsilon_{3012}}^{-1}\overbrace{(dt\wedge dx\wedge dy)}^{1}+\overbrace{\epsilon_{3021}}^{1}\overbrace{(dt\wedge dy\wedge dx)}^{-1}+\overbrace{\epsilon_{3102}}^{1}\overbrace{(dx\wedge dt\wedge dy)}^{-1} \\
&\quad +\overbrace{\epsilon_{3120}}^{-1}\overbrace{(dx\wedge dy\wedge dt)}^{1}+\overbrace{\epsilon_{3201}}^{-1}\overbrace{(dy\wedge dt\wedge dx)}^{1}+\overbrace{\epsilon_{3210}}^{1}\overbrace{(dy\wedge dx\wedge dt)}^{-1}] \\
&= \frac{1}{6}[-6(dx\wedge dy\wedge dt)] \\
&= -(dx\wedge dy\wedge dt)
\end{aligned}$$

$*(dz \wedge dt) = dx \wedge dy$

$$\begin{aligned}
*(dz\wedge dt) &= \frac{1}{4}(-2)\delta^{\alpha 3}\delta^{\beta 0}dx^{\mu}\wedge dx^{\nu} \\
&= -\frac{1}{2}\epsilon_{30\mu\nu}dx^{\mu}\wedge dx^{\nu} \\
&= -\frac{1}{2}[\overbrace{\epsilon_{3012}}^{-1}\overbrace{dx\wedge dy}^{1}+\overbrace{\epsilon_{3021}}^{1}\overbrace{dy\wedge dx}^{-1}] \\
&= -\frac{1}{2}[-(dx\wedge dy)-(dx\wedge dy)] \\
&= -\frac{1}{2}[-2(dx\wedge dy)] \\
&= dx\wedge dy
\end{aligned}$$

$*(dy \wedge dt) = dz \wedge dx$

$$\begin{aligned}
*(dy \wedge dt) &= \frac{1}{4}(-2)\delta^{\alpha 2}\delta^{\beta 0} dx^\mu \wedge dx^\nu \\
&= -\frac{1}{2}\epsilon_{20\mu\nu} dx^\mu \wedge dx^\nu \\
&= -\frac{1}{2}[\overbrace{\epsilon_{2013}}^{1}\overbrace{dx \wedge dz}^{-1} + \overbrace{\epsilon_{2031}}^{-1}\overbrace{dz \wedge dx}^{1}] \\
&= -\frac{1}{2}[-(dz \wedge dx) - (dz \wedge dx)] \\
&= -\frac{1}{2}[-2(dz \wedge dx)] \\
&= dz \wedge dx
\end{aligned}$$

$*(dx \wedge dy) = -dz \wedge dt$

$$\begin{aligned}
*(dx \wedge dy) &= \frac{1}{4}(2)\delta^{\alpha 1}\delta^{\beta 2} dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}\epsilon_{12\mu\nu} dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}[\overbrace{\epsilon_{1203}}^{1}\overbrace{dt \wedge dz}^{-1} + \overbrace{\epsilon_{1230}}^{-1}\overbrace{dz \wedge dt}^{1}] \\
&= \frac{1}{2}[-(dz \wedge dt) - (dz \wedge dt)] \\
&= \frac{1}{2}[-2(dz \wedge dt)] \\
&= -dz \wedge dt
\end{aligned}$$

$*(dy \wedge dz) = -dx \wedge dt$

$$\begin{aligned}
*(dy \wedge dt) &= \frac{1}{4}(2)\delta^{\alpha 2}\delta^{\beta 3} dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}\epsilon_{23\mu\nu} dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}[\overbrace{\epsilon_{2301}}^{1}\overbrace{dt \wedge dx}^{-1} + \overbrace{\epsilon_{2310}}^{-1}\overbrace{dx \wedge dt}^{1}] \\
&= \frac{1}{2}[-(dx \wedge dt) - (dx \wedge dt)] \\
&= \frac{1}{2}[-2(dx \wedge dt)] \\
&= -dx \wedge dt
\end{aligned}$$

$*(dz \wedge dx) = -dy \wedge dt$

$$
\begin{aligned}
*(dz \wedge dx) &= \frac{1}{4}(2)\delta^{\alpha 3}\delta^{\beta 1} dx^{\mu} \wedge dx^{\nu} \\
&= \frac{1}{2}\epsilon_{31\mu\nu} dx^{\mu} \wedge dx^{\nu} \\
&= \frac{1}{2}[\overbrace{\epsilon_{3102}}^{1}\overbrace{dt \wedge dy}^{-1} + \overbrace{\epsilon_{3120}}^{-1}\overbrace{dy \wedge dt}^{1}] \\
&= \frac{1}{2}[-(dy \wedge dt) - (dy \wedge dt)] \\
&= \frac{1}{2}[-2(dy \wedge dt)] \\
&= -dy \wedge dt
\end{aligned}
$$

Problem 43

The definition of the Field Tensor is $F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$.

Part A: $F = \omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}}$

Since $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. Let $\mu = 0,1,2,3$ and $\nu = 0,1,2,3$:

$$\begin{aligned} F &= \overbrace{\frac{1}{2}F_{00}(dx^0 \wedge dx^0)}^{0} + \frac{1}{2}F_{01}(dx^0 \wedge dx^1) + \frac{1}{2}F_{02}(dx^0 \wedge dx^2) + \frac{1}{2}F_{03}(dx^0 \wedge dx^3) \\ &+ \frac{1}{2}F_{10}(dx^1 \wedge dx^0) + \overbrace{\frac{1}{2}F_{11}(dx^1 \wedge dx^1)}^{0} + \frac{1}{2}F_{12}(dx^1 \wedge dx^2) + \frac{1}{2}F_{13}(dx^1 \wedge dx^3) \\ &+ \frac{1}{2}F_{20}(dx^2 \wedge dx^0) + \frac{1}{2}F_{21}(dx^2 \wedge dx^1) + \overbrace{\frac{1}{2}F_{22}(dx^2 \wedge dx^2)}^{0} + \frac{1}{2}F_{23}(dx^2 \wedge dx^3) \\ &+ \frac{1}{2}F_{30}(dx^3 \wedge dx^0) + \frac{1}{2}F_{31}(dx^3 \wedge dx^1) + \frac{1}{2}F_{32}(dx^3 \wedge dx^2) + \overbrace{\frac{1}{2}F_{33}(dx^3 \wedge dx^3)}^{0} \end{aligned}$$

Using the components of the $F_{\mu\nu}$ tensor and changing dx^i to dt, dx, dy, dz, F becomes:

$$\begin{aligned} F &= \frac{1}{2}(-E_1)(dt \wedge dx) + \frac{1}{2}(-E_2)(dt \wedge dy) + \frac{1}{2}(-E_3)(dt \wedge dz) \\ &+ \frac{1}{2}(E_1)(dx \wedge dt) + \frac{1}{2}(B_3)(dx \wedge dy) + \frac{1}{2}(-B_2)(dx \wedge dz) \\ &+ \frac{1}{2}(E_2)(dy \wedge dt) + \frac{1}{2}(-B_3)(dy \wedge dx) + \frac{1}{2}(B_1)(dy \wedge dz) \\ &+ \frac{1}{2}(E_3)(dz \wedge dt) + \frac{1}{2}(B_2)(dz \wedge dx) + \frac{1}{2}(-B_1)(dz \wedge dy) \end{aligned}$$

Grouping the dt terms together:

$$\begin{aligned} F &= (E_x dx \wedge dt) + (E_y dy \wedge dt) + (E_z dz \wedge dt) \\ &+ (B_x)(dy \wedge dz) + (B_y)(dz \wedge dx) + (B_z)(dx \wedge dy) \end{aligned}$$

Notice that $\omega_{\vec{A}} = A_i dx^i$ for $i = 1,2,3$. So, $\omega_{\vec{E}} = (E_x dx) + (E_y dy) + (E_z dz)$.

Also notice that $\Phi_{\vec{A}} = \frac{1}{2}A_i \epsilon_{ijk} dx^j \wedge dx^k$. So, $\Phi_{\vec{B}} = (B_x)(dy \wedge dz) + (B_y)(dz \wedge dx) + (B_z)(dx \wedge dy)$.

Therefore $F = \omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}}$.

Part B: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Given: $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$
Rearraging you get $\frac{\partial \vec{A}}{\partial t} = -\nabla V - \vec{E}$
So, $\frac{\partial A_1}{\partial t} = -\partial_1 V - E_1$ $\frac{\partial A_2}{\partial t} = -\partial_2 V - E_2$ $\frac{\partial A_3}{\partial t} = -\partial_3 V - E_3$

Given: $\vec{B} = \nabla \times \vec{A}$
Thus: $B_1 = \partial_2 A_3 - \partial_3 A_2$ $B_2 = \partial_3 A_1 - \partial_1 A_3$ $B_3 = \partial_1 A_2 - \partial_2 A_1$

Let $\mu = 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3$ in $F_{\mu\nu}$:

$$\begin{aligned}
F_{0,0} &= \partial_0 A_0 - \partial_0 A_0 = 0 \\
F_{0,1} &= \partial_0 A_1 - \partial_1 A_0 = (-\partial_1 V - E_1) - 0 = -E_1 \\
F_{0,2} &= \partial_0 A_2 - \partial_2 A_0 = (-\partial_2 V - E_2) - 0 = -E_2 \\
F_{0,3} &= \partial_0 A_3 - \partial_3 A_0 = (-\partial_3 V - E_3) - 0 = -E_3 \\
F_{1,0} &= \partial_1 A_0 - \partial_0 A_1 = 0 - (-\partial_1 V - E_1) = E_1 \\
F_{1,1} &= \partial_1 A_1 - \partial_1 A_1 = 0 \\
F_{1,2} &= \partial_1 A_2 - \partial_2 A_1 = B_3 \\
F_{1,3} &= \partial_1 A_3 - \partial_3 A_1 = -B_2 \\
F_{2,0} &= \partial_2 A_0 - \partial_0 A_2 = 0 - (-\partial_2 V - E_2) = E_2 \\
F_{2,1} &= \partial_2 A_1 - \partial_1 A_2 = -B_3 \\
F_{2,2} &= \partial_2 A_2 - \partial_2 A_2 = 0 \\
F_{2,3} &= \partial_2 A_3 - \partial_3 A_2 = B_1 \\
F_{3,0} &= \partial_3 A_0 - \partial_0 A_3 = 0 - (-\partial_3 V - E_3) = E_3 \\
F_{3,1} &= \partial_3 A_1 - \partial_1 A_3 = B_2 \\
F_{3,2} &= \partial_3 A_2 - \partial_2 A_3 = -B_1 \\
F_{3,3} &= \partial_3 A_3 - \partial_3 A_3 = 0
\end{aligned}$$

This creates the Field Tensor:

$$F_{\mu\nu} = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

Problem 44

$$\phi = \frac{1}{x^2+y^2}\left(xdy - ydx\right)$$

$$\begin{aligned}
d\phi &= d\left(\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx\right) \\
&= \partial_m\left(\frac{x}{x^2+y^2}\right)dx^m \wedge dy - \partial_m\left(\frac{y}{x^2+y^2}\right)dx^m \wedge dx \\
&= \left[\partial_x\left(\frac{x}{x^2+y^2}\right)dx + \partial_y\left(\frac{x}{x^2+y^2}\right)dy\right] \wedge dy - \left[\partial_x\left(\frac{y}{x^2+y^2}\right)dx + \partial_y\left(\frac{y}{x^2+y^2}\right)dy\right] \wedge dx \\
&= \partial_x\left(\frac{x}{x^2+y^2}\right)dx \wedge dy + \overbrace{\partial_y\left(\frac{x}{x^2+y^2}\right)dy \wedge dy}^{0} - \overbrace{\partial_y\left(\frac{y}{x^2+y^2}\right)dx \wedge dx}^{0} - \partial_y\left(\frac{y}{x^2+y^2}\right)\overbrace{dy \wedge dx}^{-1} \\
&= \partial_x\left(\frac{x}{x^2+y^2}\right)dx \wedge dy + \partial_y\left(\frac{y}{x^2+y^2}\right)dx \wedge dy \\
&= \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]dx \wedge dy \\
&= \left[\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right]dx \wedge dy \\
&= \left[\frac{2}{x^2+y^2} - \frac{2}{x^2+y^2}\right]dx \wedge dy \\
&= 0
\end{aligned}$$

Problem 45

Using 43a, we know

$$F = \omega_E \wedge dt + \Phi_B$$

Since $A = A_\mu dx^\mu$, $A_\mu = (-V, \vec{A})$

$$\begin{aligned}
dA &= d(-Vdt + \omega_A) \\
&= -dV \wedge dt + (\partial_\mu A_i)dx^\mu \wedge dx^i \\
&= -dV \wedge dt + (\partial_t A_i)dt \wedge dx^i + (\partial_j A_i)dx^j \wedge dx^i \\
&= -(\partial_k V)dx^k \wedge dt - (\partial_t A_i)dx^i \wedge dt + (\partial_j A_i)dx^j \wedge dx^i \\
&= \omega_{-\nabla V} \wedge dt + \omega_{-\partial_t A} \wedge dt + \Phi_{\nabla\times A} \\
&= \omega_{-\nabla V-\partial_t A} \wedge dt + \Phi_{\nabla\times A} \\
&= \omega_E \wedge dt + \Phi_B
\end{aligned}$$

So $F = dA \Rightarrow$ exact $\Rightarrow$ closed

Problem 46

$A = A_\mu dx^\mu$ and $A' = A + d\lambda$

$$\begin{aligned} dA &= d(A_\mu dx^\mu) \\ &= dA_\mu \wedge dx^\mu \\ &= \partial_\nu A_\mu dx^\nu \wedge dx^\mu \end{aligned}$$

$$\begin{aligned} dA' &= d(A + d\lambda) \\ &= dA + d(d\lambda) \\ &= \partial_\nu A_\mu dx^\nu \wedge dx^\mu + d^2\lambda \\ &= \partial_\nu A_\mu dx^\nu \wedge dx^\mu + 0 \\ &= \partial_\nu A_\mu dx^\nu \wedge dx^\mu \end{aligned}$$

Therefore, $dA = dA'$

Alternately, you could show this by

$$\begin{aligned} dA' &= d(A + d\lambda) \\ &= dA + d(d\lambda) \\ &= dA + d^2\lambda \\ &= dA \end{aligned}$$

46. Show that $A = A_\mu dx^\mu$ and $A' = A + d\lambda$ yield the same field tensor.

$F = dA$ and $F' = dA'$

We wish to show that $F = F'$.

$$\begin{aligned} F' &= dA' \\ &= d(A + d\lambda) \\ &= dA + \underbrace{d^2\lambda}_{=0} \\ &= dA \\ &= F \end{aligned}$$

$\therefore\ F = F'$

47. Before we discussed the Coulomb Gauge $\nabla \bullet A = 0$ and the Lorentz gauge $\partial_\mu A^\mu = 0$. Which of these gauge choices is preserved under a Lorentz transformation?

First,

Want to show that $\nabla \bullet A = 0$ does not imply $\overline{\nabla} \bullet \overline{A} = 0$ so we will show a counterexample.

We have the x-boost: $\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

We assume that $\nabla \bullet A = 0$ holds in the x-frame.

$$\overline{\nabla} \bullet \overline{A} = \overline{\partial}_i \overline{A^i} = \frac{\partial}{\partial \overline{x}}\left(\overline{A_x}\right) + \frac{\partial}{\partial \overline{y}}\left(\overline{A_y}\right) + \frac{\partial}{\partial \overline{z}}\left(\overline{A_z}\right)$$

We start by examining each term on the right-hand side.

$$\begin{aligned} \frac{\partial}{\partial \overline{x}} &= \left(\Lambda^{-1}\right)^\mu_1 \frac{\partial}{\partial x^\mu} \\ &= \left(\Lambda^{-1}\right)^0_1 \frac{\partial}{\partial t} + \left(\Lambda^{-1}\right)^1_1 \frac{\partial}{\partial x} + \left(\Lambda^{-1}\right)^2_1 \frac{\partial}{\partial y} + \left(\Lambda^{-1}\right)^3_1 \frac{\partial}{\partial z} \\ &= \gamma\beta \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x} + 0 + 0 \end{aligned}$$

$$\overline{A_\mu} = \left(\Lambda^{-1}\right)^\nu_\mu A_\nu$$
$$\overline{A_1} = \left(\Lambda^{-1}\right)^0_1 A_0 + \left(\Lambda^{-1}\right)^1_1 A_1 + \left(\Lambda^{-1}\right)^2_1 A_2 + \left(\Lambda^{-1}\right)^3_1 A_3$$
$$= \gamma\beta A_0 + \gamma A_1 + 0 + 0$$
$$= \gamma\beta A_t + \gamma A_x + 0 + 0$$

$\frac{\partial}{\partial \overline{y}}\left(\overline{A_y}\right) = \frac{\partial A_y}{\partial y}$ and $\frac{\partial}{\partial \overline{z}}\left(\overline{A_z}\right) = \frac{\partial A_z}{\partial z}$ since we are in an x-boost.

By substitutions, we have

$$\overline{\nabla} \bullet \overline{A} = \left(\gamma\beta\frac{\partial}{\partial t} + \gamma\frac{\partial}{\partial x}\right)(\gamma\beta A_t + \gamma A_x) + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= \left(\gamma\beta\frac{\partial}{\partial t}\right)(\gamma\beta A_t + \gamma A_x) + \left(\gamma\frac{\partial}{\partial x}\right)(\gamma\beta A_t + \gamma A_x) + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= \underbrace{\gamma^2\beta^2\frac{\partial A_t}{\partial t} + \gamma^2\beta\frac{\partial A_x}{\partial t} + \gamma^2 v\frac{\partial A_t}{\partial x} + \left(\gamma^2 - 1\right)\frac{\partial A_x}{\partial x}}_{\neq 0} + \underbrace{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}_{\nabla \bullet A = 0}$$

$\therefore$ $\overline{\nabla} \bullet \overline{A} \neq 0$, so the Coulomb gauge is not preserved under a Lorentz transformation.

Next,

Want to show that $\partial_\mu A^\mu = 0 \Rightarrow \overline{\partial}_\mu \overline{A}^\mu = 0$.

We know that $\overline{\partial}_\mu = \left(\Lambda^{-1}\right)^\alpha_\mu \partial_\alpha$ and $\overline{A}^\mu = (\Lambda)^\mu_\beta A^\beta$.

$$\overline{\partial}_\mu \overline{A}^\mu = \left(\Lambda^{-1}\right)^\mu_\alpha \partial_\alpha (\Lambda)^\beta_\mu A^\beta$$
$$= \left(\Lambda^{-1}\right)^\mu_\alpha (\Lambda)^\beta_\mu \partial_\alpha A^\beta$$
$$= \delta^\beta_\alpha \partial_\alpha A^\beta$$
$$= \partial_\beta A^\beta$$
$$= \partial_\mu A^\mu$$
$$= 0$$

$\therefore$ $\partial_\mu A^\mu = 0 \Rightarrow \overline{\partial}_\mu \overline{A}^\mu = 0$

So the Lorentz gauge is preserved under a Lorentz transformation.

48. Show that if a charge q is at rest with $\vec{B} = B\hat{z}$ in S then it is in constant velocity motion in S' where S' is x-boosted S.

Let us consider $S \to (t, x, y, z)$ and $S' \to (t', x', y', z')$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

In S –

$\vec{B} = B\hat{z}$ where $\vec{B} = (0, 0, B)$ and $\vec{E} = (0, 0, 0)$

Assume a charge, q, is at rest.

The force on the charge is $F = q(\vec{E} + \vec{u} \times \vec{B})$ and $\vec{u}(t = 0) = 0$.

Thus, with the equation of motion

$$F = \underbrace{\frac{1}{c^2}(F \bullet \vec{u})\vec{u}}_{=0 \ b/c \ \vec{u}=0} + m_0\gamma(\vec{u})\frac{d\vec{u}}{dt} \Rightarrow m_0\gamma(\vec{u})\frac{d\vec{u}}{dt} \Rightarrow \frac{d\vec{u}}{dt} = 0 \Rightarrow \vec{u} = k \in R \quad \& \quad k = 0$$

In S' –

$x'(t' = 0) = y'(t' = 0) = z'(t' = 0) = 0$ and $u' = (-\beta, 0, 0)$.

So, $F' = q(E' + u' \times B')$.

We need to find E' & B'.

To do so, we will use the results from a previous homework problem along with $u' = (-\beta, 0, 0)$, $\vec{B} = (0, 0, B)$ and $\vec{E} = (0, 0, 0)$:

We know $F'_{\mu\nu} = (\Lambda^{-1})^{\alpha}_{\mu}(\Lambda^{-1})^{\beta}_{\nu} F_{\alpha\beta}$ gives us

$$\begin{aligned} E_1' &= E_1 = 0 & B_1' &= B_1 = 0 \\ E_2' &= \gamma(E_2 - \beta B_3) = -\gamma\beta B & B_2' &= \gamma(B_2 + \beta E_3) = 0 \\ E_3' &= \gamma(E_3 + \beta B_2) = 0 & B_3' &= \gamma(B_3 - \beta E_2) = \gamma B \end{aligned}$$

So, $E' = (0, -\gamma\beta B, 0)$ and $B' = (0, 0, \gamma B)$.

Now plug those values into $F' = q(E' + u' \times B')$

$$\begin{aligned} F' &= q((0, -\gamma\beta B, 0) + (-\beta, 0, 0) \times (0, 0, \gamma B)) \\ &= q((0, -\gamma\beta B, 0) + (0, \gamma\beta B, 0)) \\ &= q((0, 0, 0)) \\ &= 0 \end{aligned}$$

Therefore, there is no force charge in the moving frame.

Thus,

$$F = 0 \;\Rightarrow\; \frac{d\vec{u'}}{dt'} = 0 \;\Rightarrow\; \vec{u'} = k, \;\; k \in R$$

So there is constant velocity in the moving frame.